Numerical and Analytical Studies on Charged Anisotropic Fluid Spheres in General Relativity

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Abstract A model of a charged anisotropic fluid sphere in General Relativity without using the condition \( T_{1}^{1} = 0 \) is further developed following an earlier publication of the author. An attempt to remove some of the undesirable features such as the singularities of the radial pressure at \( r = 0 \) of the older model is made. Numerical solutions of the Einstein-Maxwell’s field equations for the \( g_{00} \) component of the metric tensor is given assuming expected models of the stellar pressure and density distributions.

Keywords Anisotropic Fluid Spheres, General Relativity, Numerical Methods

1. Introduction

A model of a charged anisotropic fluid sphere was developed by the author[1] along with collaborators. In the intervening years this above publication received quite a few citations[2-4]. Owing to the demise of one of the principal authors (Prof. N.C. Rana) we had been unable to further develop this model. In reference[2] it is pointed out that our model is somewhat different from the others existing in literature in the sense that that the condition \( T_{1}^{1} = 0 \) is not enforced for example in contrast to those of Gron[5] and Ponce de Leon[6]. In the present paper we plan to develop and correct some of the undesirable features in our former publication[1] like the singularity in the radial pressure \( p_{r} \) at \( r = 0 \). We had commented then that to obtain the distribution of pressure and density that is expected to exist in real stars it may be necessary to solve the general relativistic field equations using numerical techniques. However in applying such methods one has to consider the actual mass density distribution that is expected to exist in a star like object like the Sun for example. In section 2 we give an analytical solution of the field equations in which the radial pressure is free of any singularities. It also satisfies the physical requirements like \( p_{r} > 0 \) and \( \frac{dp_{r}}{dr} < 0 \).

However the mass density distribution still has some undesirable features as we shall see later. In section 3 we take a star like object which has the same mass and radius as the sun and subject it to a numerical analysis under the assumption of a linearly falling mass density from the centre to the periphery as shown in fig. 1. The residual charge density is on the other hand maintained at an unusually high value for a normal star in order to study its effect on the metric.

Figure 1. Stellar density as a function of the radial coordinate from the centre to the surface

2. Formulation of the Problem and its Analytical Solution

The interior metric of the sphere and the independent set of Einstein-Maxwell’s field equations[1] are respectively

\[
ds^2 = e^\lambda dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 - c^2 e^\nu dt^2
\]
and

$$e^{-\lambda} = 1 - \frac{2Gm(r)}{c^2 r} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2$$  \hspace{1cm} (4)$$

The radial pressure is then obtained as

$$p_r = A \left[1 - \frac{2Gm(r)}{c^2 r} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2\right] - c^2 \rho(r)$$  \hspace{1cm} (8)$$

Now to make \( p_r = 0 \) at \( r = a \) the constant \( A \) will be

$$A = c^2 \rho(a) \left[1 - \frac{2Gm(a)}{c^2 a} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2\right]^{-1}$$  \hspace{1cm} (9)$$

The tangential pressure in terms of the above quantities is obtained from Eq. (3) and is expressed as

$$p_{\perp} = p_r + \frac{r}{4} \left(\frac{c^2 \rho(r) + p_r \nu' + 2p_r' - (4\pi\rho_0 c)^2}{2\pi r^3}\right)$$

The conditions that the radial pressure must be positive and its gradient negative that is \( p_r > 0 \) and \( \frac{dp_r}{dr} < 0 \) is expressed as

$$\frac{\rho(a)}{1 - \frac{2Gm(a)}{c^2 a} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2} > \frac{\rho(r)}{1 - \frac{2Gm(r)}{c^2 r} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2}$$

and

$$\frac{dp_r}{dr} > \frac{\rho(a)}{1 - \frac{2Gm(a)}{c^2 a} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2} - \frac{2Gm'(r)}{c^2 r}$$

For normal density stars these two will be satisfied only if \( \rho(r) \) is a constant or an increasing function of the radial coordinate. Usually in real stars the density falls from the core to the periphery and only in rare cases it may be considered to be approximately constant so the application of the above results to such bodies is quite remote. We give a numerical solution of Eq. (6) below for a more realistic model.

### 3. Numerical Solution of the Einstein Equations for a Sun Like Object with Some Excess Charge and Anisotropic Pressure

The problem we have to solve is that given the form of \( e^{-\lambda} \) by Eq. (4) we must obtain a solution for \( \nu' \) from the differential equation expressed by Eq. (6). Thus in order to numerically integrate Eq. (6) expressed as

$$\nu = \int F(R) dR + \text{cons tan} t$$  \hspace{1cm} (10)$$

with the function \( F(R) \) given by

$$F(R) = \frac{8\pi G \rho_0 (R) + 2Gm(R)}{1 - \frac{2Gm(R)}{c^2 R} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2}$$

and

$$e^{-\lambda} = 1 - \frac{2Gm(r)}{c^2 r} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2$$  \hspace{1cm} (7)$$

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$$\frac{dp_r}{dr} > \frac{\rho(a)}{1 - \frac{2Gm(a)}{c^2 a} + \frac{G}{c^2} \left(\frac{4\pi\rho_0 c}{a}\right)^2} - \frac{2Gm'(r)}{c^2 r}$$

For normal density stars these two will be satisfied only if \( \rho(r) \) is a constant or an increasing function of the radial coordinate. Usually in real stars the density falls from the core to the periphery and only in rare cases it may be considered to be approximately constant so the application of the above results to such bodies is quite remote. We give a numerical solution of Eq. (6) below for a more realistic model.
Figure 2. The radial pressure plotted as a function of the radial coordinate from the core to the stellar surface.

Figure 3. The $g_{00}$ component of the metric tensor plotted as a function of the radial coordinate from the centre to the stellar surface for the model developed in this paper.

We have to give the expected values of $p(r)$ and $m(r)$. The nature of $\rho(r)$ is as already stated given by fig. 1 in e.s.u. (cgs) units roughly corresponding to a total mass of the Sun when integrated out from zero to a value of the solar radius $a = 696 \times 10^8$ cm. The function $p(r)$ is shown in fig. 2 falling from a maximum value at the centre which is roughly the expected pressure near the core of the Sun to zero on the surface.

In equation (10) the value of the constant that is $\nu(0)$ has to be adjusted so that $e^{\nu(a)}$ should be equal to the Reissner–Nordstrom value which is $e^{\lambda(a)}$. Also one has to provide the value of $G_4(4\pi\rho_{oc})^2$. Normal stars like the Sun is on the whole charge neutral. Nevertheless certain types of stellar models do have some residual charge. We have set the value of $\rho_{oc}$ to be somewhat abnormally high to study its effect and have consequently set $0.2Gm(a) = \frac{G}{c^4} (4\pi\rho_{oc})^2$. The value of $e^{\nu(r)}$ is plotted as a function of $r$ in fig. 3. In fig. 4 for the purpose of comparison $e^{\nu(r)}$ is plotted for a charge neutral model[7] using Eq. (11) of this reference. The value of $n$ for the typical mass and radius of the Sun turns out to be $2.122 \times 10^{-6}$. In connection with the subject of stellar models in general relativity which has direct relevance to the present study we draw attention to two more publications[8],[9].

REFERENCES


