

# Entanglement between Two Tavis-Cummings Systems with $N=2$

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**Abstract** We consider two non interacting two-atoms Tavis-Cummings systems and examine the entanglement among four two-state atoms. For our purpose, we assume that one atom from each system are entangled initially and we obtain time-dependent concurrence among atoms. There are six pairwise concurrences among atoms and in this paper, we focus on remote atoms. We show that initially non-entangled atoms  $A_2$  and  $B_2$  becomes entangled by passing time, the process which may be interpreted as entanglement transfer between the cavities.

**Keywords** Entanglement, Tavis-Cummings Model, Concurrence

## 1. Introduction

Entanglement is one of the most fascinating aspect of quantum mechanics and is a defining feature that makes fundamental distinctions between quantum and classical Physics. Two quantum systems A and B are entangled when values of certain properties of system A are correlated with the same properties of system B [1]. Quantum entanglement has been viewed as an essential resource for quantum information process, and a great deal of effort has been devoted to study and characterize the entanglement. Cavity quantum electrodynamics (QED) techniques has been recognized as a promising candidate for the physical realization of quantum information processing [2].

In this paper we consider two non interacting two-atoms Tavis-Cummings systems and examine the entanglement among four two-states atoms by obtaining the concurrences between each two atoms. The method which we have used is like the method that is used in ref [3]. For this purpose we first introduce the Tavis-Cummings model and then obtain the eigenvalues and eigenstates of the two-atoms Tavis-Cummings system in section 2. These eigenstates and eigenvalues are used for calculating the time evolution of entanglement. We introduce the concept of concurrence for measuring the entanglement in section 3 and finally in section 4, we analyze the entanglement among atoms in detail by calculating the reduced density matrix of each two two-levels atoms and give some concludes in section 5.

## 2. The Tavis-Cummings Model

The Tavis-Cummings model (TCM) describes the simplest fundamental interaction between a single mode of the quantized electromagnetic field and a collection of  $N$  two-level atoms under the rotating wave approximation (RWA) condition [4] - [11]. The Hamiltonian of this model is written as

$$H_{TC} = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega \sum_{i=1}^N \sigma_3^{(i)} + \hbar\lambda \sum_{i=1}^N (\sigma_+^{(i)} a + \sigma_-^{(i)} a^\dagger) \quad (1)$$

For the case of two-atom ( $N=2$ )

$$\begin{aligned} H &= H_0 + H_{int} \\ &= \hbar\omega \left( a^\dagger a + \frac{1}{2} \sigma_3^{(1)} + \frac{1}{2} \sigma_3^{(2)} \right) \\ &\quad + \hbar\lambda \left[ (\sigma_-^{(1)} + \sigma_-^{(2)}) a^\dagger + (\sigma_+^{(1)} + \sigma_+^{(2)}) a \right] \quad (2) \end{aligned}$$

Where  $a$  and  $a^\dagger$  are the annihilation and creation field operators.

Each initial state in this system is coupled to three states. For example if the initial state is  $|i\rangle = |e_1\rangle|e_2\rangle|n\rangle$  then this state is coupled with the states

$$\begin{aligned} |f_1\rangle &= |e_1\rangle|g_2\rangle|n+1\rangle \\ |f_2\rangle &= |g_1\rangle|e_2\rangle|n+1\rangle \\ |f_3\rangle &= |g_1\rangle|g_2\rangle|n+2\rangle \quad (3) \end{aligned}$$

Writing the Hamiltonian in these states, we obtain

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$$H^{(n)} = \begin{pmatrix} \hbar\omega(n+1) & \hbar\lambda\sqrt{n+2} & \hbar\lambda\sqrt{n+2} & 0 \\ \hbar\lambda\sqrt{n+2} & \hbar\omega(n+1) & 0 & \hbar\lambda\sqrt{n+1} \\ \hbar\lambda\sqrt{n+2} & 0 & \hbar\omega(n+1) & \hbar\lambda\sqrt{n+1} \\ 0 & \hbar\lambda\sqrt{n+1} & \hbar\lambda\sqrt{n+1} & \hbar\omega(n+1) \end{pmatrix} \quad (4)$$

Then the eigenvalues are

$$\begin{aligned} E_{1n} &= \hbar\omega(n+1) \\ E_{2n} &= \hbar\omega(n+1) \\ E_{3n} &= \hbar\omega(n+1) + \hbar\lambda\sqrt{2(2n+3)} \\ E_{4n} &= \hbar\omega(n+1) - \hbar\lambda\sqrt{2(2n+3)} \end{aligned} \quad (5)$$

and the correlated eigenstates are

$$\begin{aligned} |\psi_{1n}\rangle &= \sqrt{\frac{n+1}{2n+3}} |g_1g_2; n+2\rangle - \sqrt{\frac{n+2}{2n+3}} |e_1e_2; n\rangle \\ |\psi_{2n}\rangle &= \frac{1}{\sqrt{2}} |g_1e_2; n+1\rangle - \frac{1}{\sqrt{2}} |e_1g_2; n+1\rangle \\ |\psi_{3n}\rangle &= \sqrt{\frac{n+2}{4n+6}} |g_1g_2; n+2\rangle + \frac{1}{2} |g_1e_2; n+1\rangle \\ &\quad + \frac{1}{2} |e_1g_2; n+1\rangle + \sqrt{\frac{n+1}{4n+6}} |e_1e_2; n\rangle \\ |\psi_{4n}\rangle &= \sqrt{\frac{n+2}{4n+6}} |g_1g_2; n+2\rangle - \frac{1}{2} |g_1e_2; n+1\rangle \\ &\quad - \frac{1}{2} |e_1g_2; n+1\rangle + \sqrt{\frac{n+1}{4n+6}} |e_1e_2; n\rangle \end{aligned}$$

### 3. Measure of Entanglement

To measure the entanglement, we need to introduce a convenient concept. We will adopt the Woote's concurrence[13] as our measure:

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\} \quad (7)$$

where the quantities  $\lambda_i$  are the eigenvalues in decreasing order of the matrix

$$\xi = \rho(\sigma_y \otimes \sigma_y) \rho^*(\sigma_y \otimes \sigma_y) \quad (8)$$

where  $\rho^*$  denotes the complex conjugation of  $\rho$  in the standard basis and  $\sigma_y$  is the Pauli matrix.

We may pay attention to this point that although the matrix  $\xi$  is not Hermitian, the eigenvalues of this matrix are real and nonnegative.

Note that  $1 \geq C \geq 0$ , where  $C = 0$  indicates zero entanglement and  $C = 1$  means maximal pure state entanglement.

Reduction to a two-qubit form, will yield to a two-qubit mixed state always having the form:

$$\rho = \begin{pmatrix} a & 0 & 0 & z \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ z^* & 0 & 0 & d \end{pmatrix} \quad (9)$$

where  $a + b + c + d = 1$ . The concurrence for this matrix is easily found to be

$$C = 2 \max\{0, |z| - \sqrt{bc}\} \quad (10)$$

### 4. Entanglement Dynamics among Atoms in Two Tavis-Cummings Systems with N=2

Consider two two-atoms Tavis-Cummings systems A and B where A consists of two atoms  $A_1, A_2$  and B consists of two atoms  $B_1, B_2$ . We assume that initially an atom of system A,  $A_1$ , is entangled with an atom of system B,  $B_1$

Now, we are going to investigate the entanglement among atoms in this two Tavis-Cummings systems by calculating the concurrence among atoms. The total initial state has three parts:

i) The entangled atoms: This initial part can be written in terms of Bell states. We denote this part as follows:

$$|\Phi_{A_1B_1}\rangle = \cos\alpha |e_{1A}e_{1B}\rangle + \sin\alpha |g_{1A}g_{1B}\rangle \quad (11)$$

ii) The two atoms which are not entangled initially. We can write this initial part as below:

$$\begin{aligned} |\Phi_{A_2B_2}(0)\rangle &= \alpha_1 |e_{2A}e_{2B}\rangle + \alpha_2 |e_{2A}g_{2B}\rangle \\ &\quad + \alpha_3 |g_{2A}e_{2B}\rangle + \alpha_4 |g_{2A}g_{2B}\rangle \end{aligned} \quad (12)$$

To make sure that these atoms are not entangled initially, we must have

$$\alpha_1\alpha_4 = \alpha_2\alpha_3 \quad (13)$$

iii) The cavities Initial states:

$$|\Phi_F(0)\rangle = |n\rangle_A \otimes |m\rangle_B \quad (14)$$

where n (m) is the number of photons in cavities A (B) at  $t=0$ .

Thus the total initial state is

$$\begin{aligned} |\Phi(0)\rangle &= (\cos\alpha |e_{1A}e_{1B}\rangle + \sin\alpha |g_{1A}g_{1B}\rangle) \\ &\quad \otimes (\alpha_1 |e_{2A}e_{12B}\rangle + \alpha_2 |e_{2A}g_{2B}\rangle) \\ &\quad + \alpha_3 |g_{2A}e_{2B}\rangle + \alpha_4 |g_{2A}g_{2B}\rangle \end{aligned} \quad (15)$$

which can be written as

$$\begin{aligned} |\Phi(0)\rangle &= \cos\alpha [\alpha_1 |e_1e_2; n\rangle_A \otimes |e_1e_2; m\rangle_B \\ &\quad + \alpha_2 |e_1e_2; n\rangle_A \otimes |e_1g_2; m\rangle_B \\ &\quad + \alpha_3 |e_1g_2; n\rangle_A \otimes |e_1e_2; m\rangle_B \\ &\quad + \alpha_4 |e_1g_2; n\rangle_A \otimes |e_1g_2; m\rangle_B] \end{aligned}$$

$$\begin{aligned}
 & + \sin \alpha [\alpha_1 |g_1 e_2; n\rangle_A \otimes |g_1 e_2; m\rangle_B \\
 & + \alpha_2 |g_1 e_2; n\rangle_A \otimes |g_1 g_2; m\rangle_B \\
 & + \alpha_3 |g_1 g_2; n\rangle_A \otimes |g_1 e_2; m\rangle_B \\
 & + \alpha_4 |g_1 g_2; n\rangle_A \otimes |g_1 g_2; m\rangle_B] \quad (16)
 \end{aligned}$$

To simplify our calculations, we suppose that the non entangled atoms are initially in excited states ( $\alpha_1 = 1$ ,  $\alpha_2 = \alpha_3 = \alpha_4 = 0$ ) and the number of photons in two cavities are equal ( $n=m$ ). By applying these conditions, the initial state becomes

$$\begin{aligned}
 |\Phi(0)\rangle = & \cos \alpha |e_1 e_2; n\rangle_A \otimes |e_1 e_2; n\rangle_B \\
 & + \sin \alpha |g_1 e_2; n\rangle_A \otimes |g_1 e_2; n\rangle_B \quad (17)
 \end{aligned}$$

To prepare for the time evolution we express these states in terms of the eigenstates given in (6), and obtain

$$\begin{aligned}
 |e_1 e_2; n\rangle & = a_{1n} |\psi_{1n}\rangle + a_{3n} |\psi_{3n}\rangle + a_{4n} |\psi_{4n}\rangle \\
 |g_1 g_2; n\rangle & = b_{1n} |\psi_{1(n-2)}\rangle + b_{3n} |\psi_{3(n-2)}\rangle + b_{4n} |\psi_{4(n-2)}\rangle \\
 |g_1 e_2; n\rangle & = c_{2n} |\psi_{2(n-1)}\rangle + c_{3n} |\psi_{3(n-1)}\rangle + c_{4n} |\psi_{4(n-1)}\rangle \\
 |e_1 g_2; n\rangle & = d_{2n} |\psi_{2(n-1)}\rangle + d_{3n} |\psi_{3(n-1)}\rangle + d_{4n} |\psi_{4(n-1)}\rangle
 \end{aligned} \quad (18)$$

where

$$\begin{aligned}
 a_{1n} & = -\sqrt{\frac{n+2}{2n+3}} \quad a_{3n} = a_{4n} = \sqrt{\frac{n+1}{4n+6}} \\
 b_{1n} & = \sqrt{\frac{n-1}{2n-1}} \quad b_{3n} = b_{4n} = \sqrt{\frac{n}{4n-2}} \\
 c_{2n} & = \frac{1}{\sqrt{2}} \quad c_{3n} = \frac{1}{2} \quad c_{4n} = -\frac{1}{2} \\
 d_{2n} & = -\frac{1}{\sqrt{2}} \quad d_{3n} = \frac{1}{2} \quad d_{4n} = -\frac{1}{2} \quad (19)
 \end{aligned}$$

Thus, the initial state is

$$\begin{aligned}
 |\Phi(0)\rangle = & \cos \alpha (a_{1n} |\psi_{1n}\rangle_A + a_{3n} |\psi_{3n}\rangle_A + a_{4n} |\psi_{4n}\rangle_A) \\
 & \otimes (a_{1n} |\psi_{1n}\rangle_B + a_{3n} |\psi_{3n}\rangle_B + a_{4n} |\psi_{4n}\rangle_B) \\
 & + \sin \alpha (c_{2n} |\psi_{2(n-1)}\rangle_A + c_{3n} |\psi_{3(n-1)}\rangle_A + c_{4n} |\psi_{4(n-1)}\rangle_A) \\
 & \otimes (c_{2n} |\psi_{2(n-1)}\rangle_B + c_{3n} |\psi_{3(n-1)}\rangle_B + c_{4n} |\psi_{4(n-1)}\rangle_B) \quad (20)
 \end{aligned}$$

Since the time evolution of these eigenstates is specified we can transfer these time evolutions to the coefficients. Therefore

$$\begin{aligned}
 |\Phi(t)\rangle = & \cos \alpha (a_{1n}(t) |\psi_{1n}\rangle_A + a_{3n}(t) |\psi_{3n}\rangle_A \\
 & + a_{4n}(t) |\psi_{4n}\rangle_A) \\
 & \otimes (a_{1n}(t) |\psi_{1n}\rangle_B + a_{3n}(t) |\psi_{3n}\rangle_B + a_{4n}(t) |\psi_{4n}\rangle_B) \\
 & + \sin \alpha (c_{2n}(t) |\psi_{2(n-1)}\rangle_A + c_{3n}(t) |\psi_{3(n-1)}\rangle_{AA}
 \end{aligned}$$

$$\begin{aligned}
 & + c_{4n}(t) |\psi_{4(n-1)}\rangle) \\
 & \otimes (c_{2n}(t) |\psi_{2(n-1)}\rangle_B + c_{3n}(t) |\psi_{3(n-1)}\rangle_B + c_{4n}(t) |\psi_{4(n-1)}\rangle_B) \quad (21)
 \end{aligned}$$

In order to take traces over individual atoms or cavities, we need to revert to the bare bases and this leads to

$$\begin{aligned}
 |\Phi(t)\rangle = & [(\beta_{1n} a_{1n}(t) + \frac{1}{\sqrt{2}} \beta_{2n} (a_{3n}(t) + a_{4n}(t))) \\
 & \times \cos \alpha |g_1 g_2; n+2\rangle_A \\
 & + (-\beta_{2n} a_{1n}(t) + \frac{1}{\sqrt{2}} \beta_{1n} (a_{3n}(t) + a_{4n}(t))) \\
 & \times \cos \alpha |e_1 e_2; n\rangle_A \\
 & + \frac{1}{2} (a_{3n}(t) - a_{4n}(t)) \cos \alpha |g_1 e_2; n+1\rangle_A \\
 & + \frac{1}{2} (a_{3n}(t) - a_{4n}(t)) \cos \alpha |g_1 e_2; n+1\rangle_A \\
 & \otimes [(\beta_{1n} a_{1n}(t) + \frac{1}{\sqrt{2}} \beta_{2n} (a_{3n}(t) + a_{4n}(t))) \\
 & \times \cos \alpha |g_1 g_2; n+2\rangle_B \\
 & + (-\beta_{2n} a_{1n}(t) + \frac{1}{\sqrt{2}} \beta_{1n} (a_{3n}(t) + a_{4n}(t))) \\
 & \times \cos \alpha |e_1 e_2; n\rangle_B \\
 & + \frac{1}{2} (a_{3n}(t) - a_{4n}(t)) \cos \alpha |g_1 e_2; n+1\rangle_B \\
 & + \frac{1}{2} (a_{3n}(t) - a_{4n}(t)) \cos \alpha |g_1 e_2; n+1\rangle_B] \\
 & + [(\frac{1}{\sqrt{2}} c_{2n}(t) + \frac{1}{2} c_{3n}(t) - \frac{1}{2} c_{4n}(t)) \sin \alpha |g_1 e_2; n\rangle_A \\
 & + (-\frac{1}{\sqrt{2}} c_{2n}(t) + \frac{1}{2} c_{3n}(t) - \frac{1}{2} c_{4n}(t)) \sin \alpha |g_1 e_2; n\rangle_A \\
 & + \frac{\gamma_{2n}}{\sqrt{2}} (c_{3n}(t) + c_{4n}(t)) \sin \alpha |g_1 g_2; n+1\rangle_A \\
 & + \frac{\gamma_{1n}}{\sqrt{2}} (c_{3n}(t) + c_{4n}(t)) \sin \alpha |e_1 e_2; n-1\rangle_A] \\
 & \otimes [(\frac{1}{\sqrt{2}} c_{2n}(t) + \frac{1}{2} c_{3n}(t) - \frac{1}{2} c_{4n}(t)) \sin \alpha |g_1 e_2; n\rangle_B \\
 & + (-\frac{1}{\sqrt{2}} c_{2n}(t) + \frac{1}{2} c_{3n}(t) - \frac{1}{2} c_{4n}(t)) \sin \alpha |g_1 e_2; n\rangle_B \\
 & + \frac{\gamma_{2n}}{\sqrt{2}} (c_{3n}(t) + c_{4n}(t)) \sin \alpha |g_1 g_2; n+1\rangle_B \\
 & + \frac{\gamma_{1n}}{\sqrt{2}} (c_{3n}(t) + c_{4n}(t)) \sin \alpha |e_1 e_2; n-1\rangle_B] \quad (22)
 \end{aligned}$$

where

$$\begin{aligned}\beta_{1n} &= \sqrt{\frac{n+1}{2n+3}} \\ \beta_{2n} &= \sqrt{\frac{n+2}{2n+3}} \\ \gamma_{1n} &= \sqrt{\frac{n}{2n+1}} \\ \gamma_{2n} &= \sqrt{\frac{n+1}{2n+1}}\end{aligned}\quad (23)$$

For simplicity, we define  $\rho^{atoms}$  by trace out from density operator on the photonic states

$$\begin{aligned}\rho^{atoms} &= \langle n+2; n+2 | \Phi(t) \rangle \langle \Phi(t) | n+2; n+2 \rangle \\ &+ \langle n+2; n+1 | \Phi(t) \rangle \langle \Phi(t) | n+2; n+1 \rangle \\ &+ \langle n+2; n | \Phi(t) \rangle \langle \Phi(t) | n+2; n \rangle \\ &+ \langle n+2; n-1 | \Phi(t) \rangle \langle \Phi(t) | n+2; n-1 \rangle \\ &+ \langle n+1; n+2 | \Phi(t) \rangle \langle \Phi(t) | n+1; n+2 \rangle \\ &+ \langle n+1; n+1 | \Phi(t) \rangle \langle \Phi(t) | n+1; n+1 \rangle \\ &+ \langle n+1; n | \Phi(t) \rangle \langle \Phi(t) | n+1; n \rangle \\ &+ \langle n+1; n-1 | \Phi(t) \rangle \langle \Phi(t) | n+1; n-1 \rangle \\ &+ \langle n; n+2 | \Phi(t) \rangle \langle \Phi(t) | n; n+2 \rangle \\ &+ \langle n; n+1 | \Phi(t) \rangle \langle \Phi(t) | n; n+1 \rangle \\ &+ \langle n; n | \Phi(t) \rangle \langle \Phi(t) | n; n \rangle \\ &+ \langle n; n-1 | \Phi(t) \rangle \langle \Phi(t) | n; n-1 \rangle \\ &+ \langle n-1; n+2 | \Phi(t) \rangle \langle \Phi(t) | n-1; n+2 \rangle \\ &+ \langle n-1; n+1 | \Phi(t) \rangle \langle \Phi(t) | n-1; n+1 \rangle \\ &+ \langle n-1; n | \Phi(t) \rangle \langle \Phi(t) | n-1; n \rangle \\ &+ \langle n-1; n-1 | \Phi(t) \rangle \langle \Phi(t) | n-1; n-1 \rangle\end{aligned}\quad (24)$$

Now, by using  $\rho^{atoms}$  we can obtain the reduced density matrix between each two atoms.

We first focus on  $C^{A_1 B_1}$ . This requires that we trace out  $A_2$  and  $B_2$  atom states, in order to get the two-qubit mixed state needed for calculation at  $A_1 B_1$  concurrence.

The  $\rho^{A_1 B_1}$  becomes

$$\begin{aligned}\rho^{A_1 B_1} &= \langle e_{2A}, e_{2B} | \rho^{atoms} | e_{2A}, e_{2B} \rangle \\ &+ \langle e_{2A}, g_{2B} | \rho^{atoms} | e_{2A}, g_{2B} \rangle \\ &+ \langle g_{2A}, e_{2B} | \rho^{atoms} | g_{2A}, e_{2B} \rangle \\ &+ \langle g_{2A}, g_{2B} | \rho^{atoms} | g_{2A}, g_{2B} \rangle\end{aligned}\quad (25)$$

We see that the  $A_1 B_1$  mixed state has the form

$$\rho^{A_1 B_1} = \begin{pmatrix} a & 0 & 0 & z \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ z^* & 0 & 0 & d \end{pmatrix}\quad (26)$$

for which the concurrence has the stated form

$$C^{A_1 B_1} = 2 \max \{0, |z| - \sqrt{bc}\}\quad (27)$$

with

$$\begin{aligned}z &= (2x_6 x_7^* + x_{12} x_{13}^* + x_4 x_5^*) \sin \alpha \cos \alpha \\ b = c &= (|x_2|^2 + |x_3|^2 + |x_4|^2 + |x_6|^2) \cos^2 \alpha \\ &+ (|x_8|^2 + |x_9|^2 + |x_{12}|^2 + |x_{14}|^2) \sin^2 \alpha\end{aligned}\quad (28)$$

The time dependent coefficients are

$$\begin{aligned}x_1 &= \left( \beta_1 a_1(t) + \frac{1}{\sqrt{2}} \beta_2 (a_3(t) + a_4(t)) \right)^2 \\ x_2 &= \frac{1}{2} \left( \beta_1 a_1(t) + \frac{1}{\sqrt{2}} \beta_2 (a_3(t) + a_4(t)) \right) (a_3(t) - a_4(t)) \\ x_3 &= \frac{1}{2} \left( \beta_1 a_1(t) + \frac{1}{\sqrt{2}} \beta_2 (a_3(t) + a_4(t)) \right) \\ &\times \left( -\beta_2 a_1(t) + \frac{1}{\sqrt{2}} \beta_1 (a_3(t) + a_4(t)) \right) \\ x_4 &= \frac{1}{4} (a_3(t) - a_4(t))^2 \\ x_5 &= \frac{\gamma_2^2}{2} (c_3(t) + c_4(t))^2 \\ x_6 &= \frac{1}{2} (a_3(t) - a_4(t)) \\ &\times \left( -\beta_2 a_1(t) + \frac{1}{\sqrt{2}} \beta_1 (a_3(t) + a_4(t)) \right) \\ x_7 &= \frac{\gamma_2}{\sqrt{2}} (c_3(t) + c_4(t)) \left( \frac{1}{\sqrt{2}} c_2(t) + \frac{1}{2} c_3(t) - \frac{1}{2} c_4(t) \right) \\ x_8 &= \frac{\gamma_2}{\sqrt{2}} (c_3(t) + c_4(t)) \left( -\frac{1}{\sqrt{2}} c_2(t) + \frac{1}{2} c_3(t) - \frac{1}{2} c_4(t) \right) \\ x_9 &= \frac{1}{2} \gamma_1 \gamma_2 (c_3(t) + c_4(t))^2 \\ x_{10} &= \left( -\beta_2 a_1(t) + \frac{1}{\sqrt{2}} \beta_1 (a_3(t) + a_4(t)) \right)^2 \\ x_{11} &= \left( \frac{1}{\sqrt{2}} c_2(t) + \frac{1}{2} c_3(t) - \frac{1}{2} c_4(t) \right)^2\end{aligned}$$

$$\begin{aligned}
 x_{12} &= \left( \frac{1}{\sqrt{2}}c_2(t) + \frac{1}{2}c_3(t) - \frac{1}{2}c_4(t) \right) \\
 &\times \left( -\frac{1}{\sqrt{2}}c_2(t) + \frac{1}{2}c_3(t) - \frac{1}{2}c_4(t) \right) \\
 x_{13} &= \left( -\frac{1}{\sqrt{2}}c_2(t) + \frac{1}{2}c_3(t) - \frac{1}{2}c_4(t) \right)^2 \\
 x_{14} &= \frac{\gamma_1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}}c_2(t) + \frac{1}{2}c_3(t) - \frac{1}{2}c_4(t) \right) (c_3(t) + c_4(t)) \\
 x_{15} &= \frac{\gamma_1}{\sqrt{2}} \left( -\frac{1}{\sqrt{2}}c_2(t) + \frac{1}{2}c_3(t) - \frac{1}{2}c_4(t) \right) (c_3(t) + c_4(t)) \\
 x_{16} &= \frac{\gamma_1^2}{2} (c_3(t) + c_4(t)) \tag{29}
 \end{aligned}$$

Figures 1-3 shows the concurrence for  $A_1B_1$  mixed state. In these graphs  $\omega = 0.5, \lambda \approx 0.27$  and  $n = 2$ .

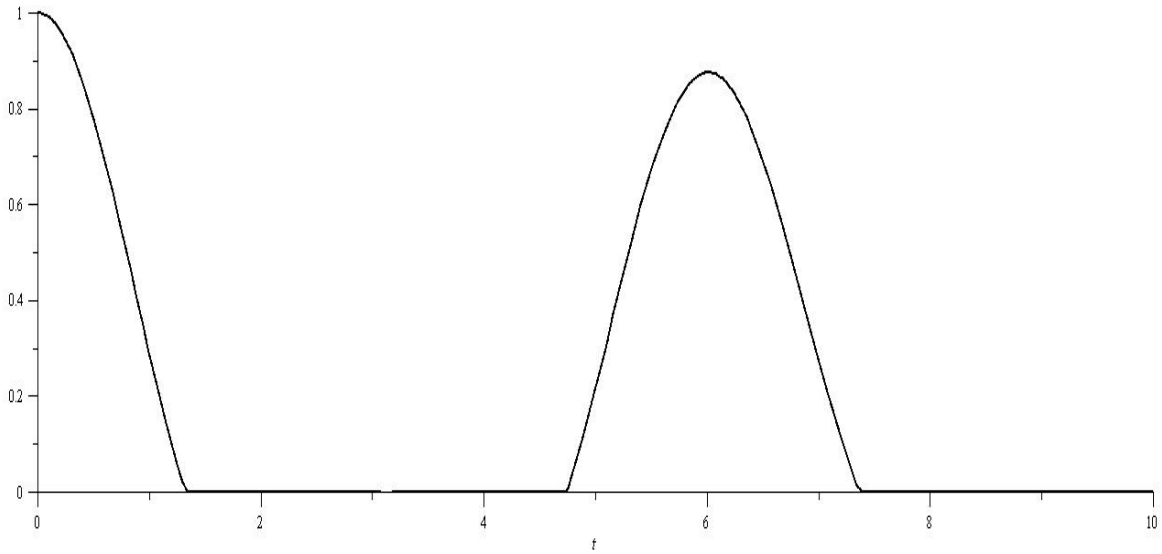


Figure 1. plot of  $A_1B_1$  concurrence for  $\alpha = \frac{\pi}{4}$

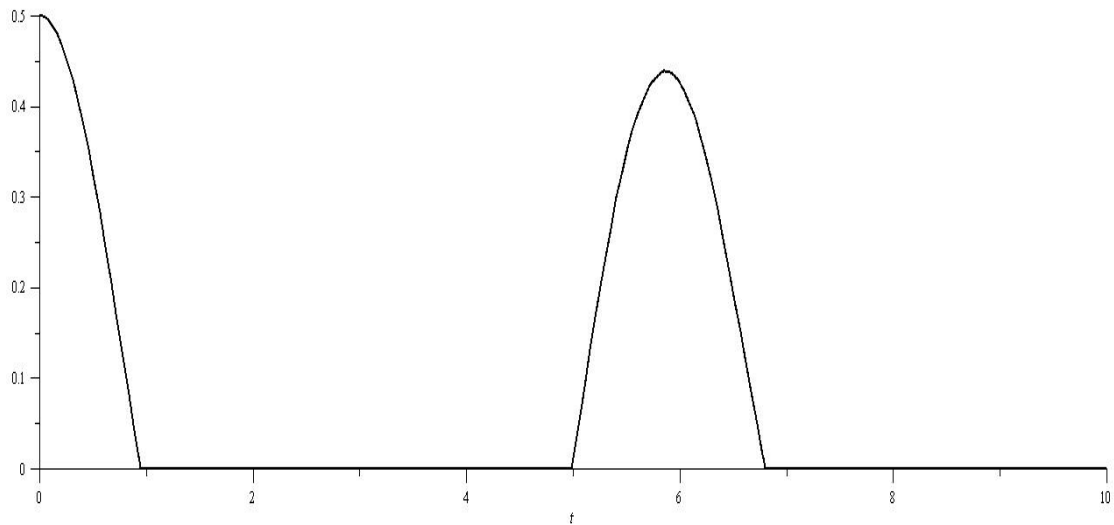


Figure 2. plot of  $A_1B_1$  concurrence for  $\alpha = \frac{\pi}{12}$

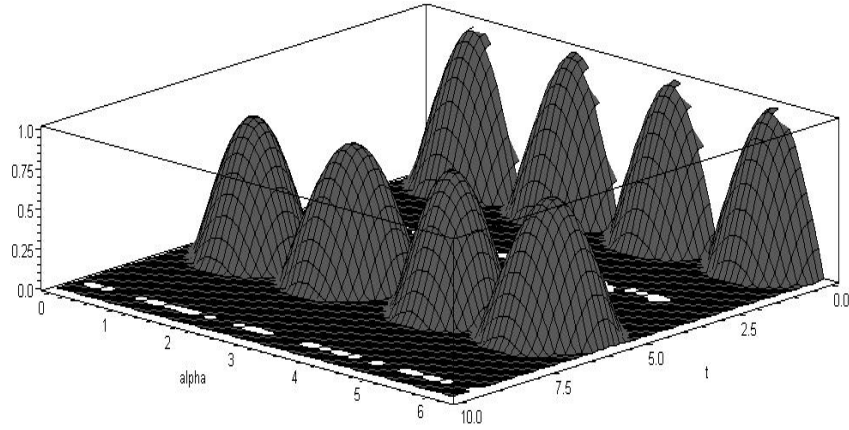


Figure 3. Surface plot of the concurrence

For concurrence calculation between  $A_2$  and  $B_2$ :

$$\begin{aligned} \rho^{A_2 B_2} = & \langle e_{1A}, e_{1B} | \rho^{atoms} | e_{1A}, e_{1B} \rangle \\ & + \langle e_{1A}, g_{1B} | \rho^{atoms} | e_{1A}, g_{1B} \rangle \\ & + \langle g_{1A}, e_{1B} | \rho^{atoms} | g_{1A}, e_{1B} \rangle \\ & + \langle g_{1A}, g_{1B} | \rho^{atoms} | g_{1A}, g_{1B} \rangle \end{aligned} \quad (30)$$

and we find

$$\begin{aligned} z = & (x_4 x_5^* + x_{10} x_{13}^* + 2x_6 x_8^*) \sin \alpha \cos \alpha \\ b = & (|x_2|^2 + |x_3|^2 + |x_4|^2 + |x_8|^2) \cos^2 \alpha \\ & + (|x_8|^2 + |x_9|^2 + |x_{12}|^2 + |x_{14}|^2) \sin^2 \alpha \\ c = & (|x_2|^2 + |x_3|^2 + |x_4|^2 + |x_6|^2) \cos^2 \alpha \\ & + (|x_7|^2 + |x_9|^2 + |x_{14}|^2 + |x_{15}|^2) \sin^2 \alpha \end{aligned} \quad (31)$$

Figures 4 and 5 show the plots for this concurrence for  $\alpha = \frac{\pi}{4}$  and  $\alpha = \frac{\pi}{12}$  at the first 100 seconds.

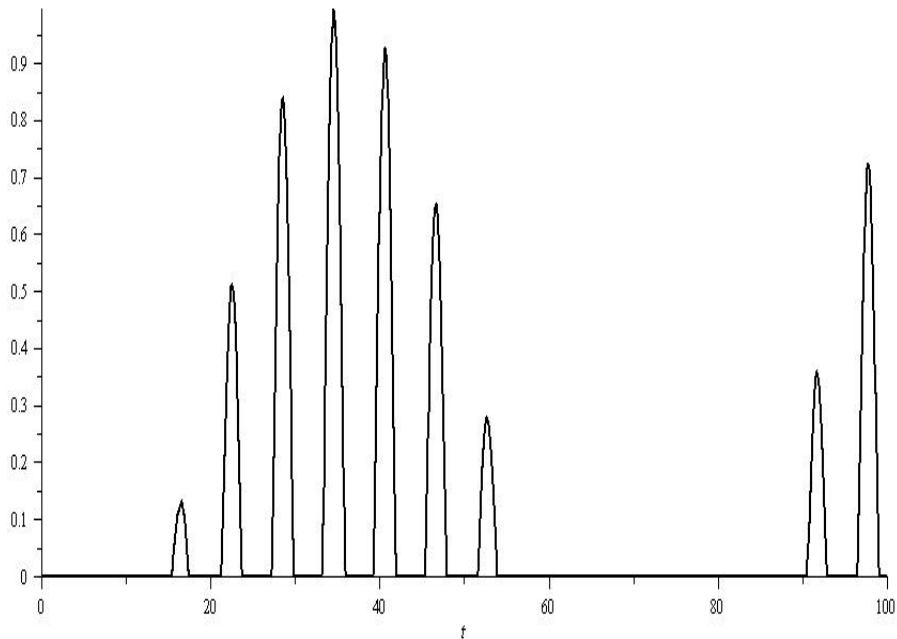


Figure 4. plot of  $A_2 B_2$  concurrence for  $\alpha = \frac{\pi}{4}$

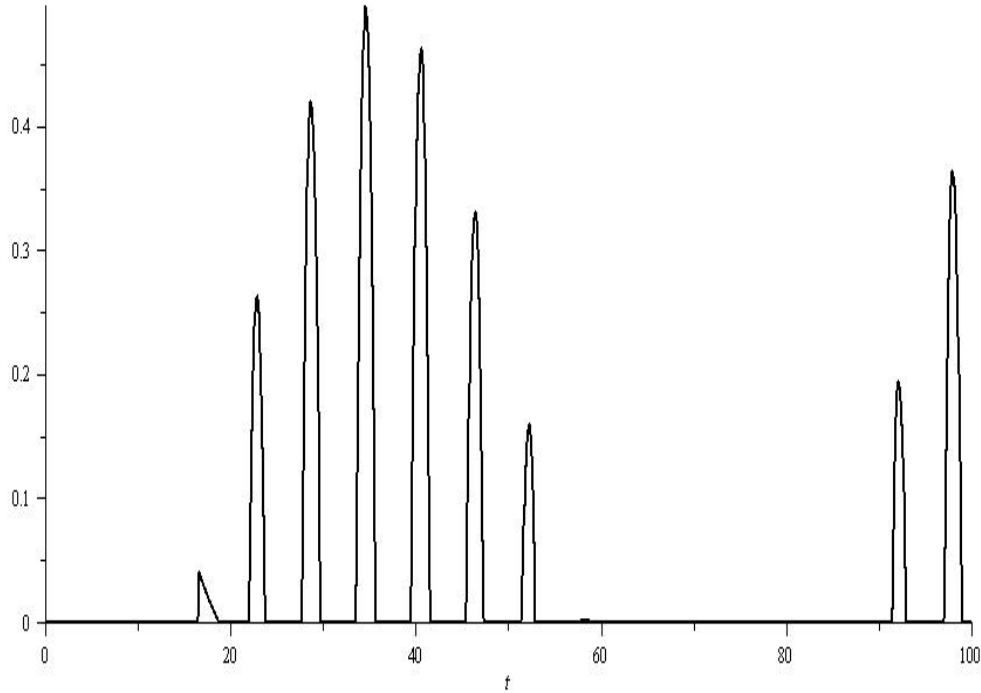


Figure 5. plot of  $A_2B_2$  concurrence for  $\alpha = \frac{\pi}{12}$

Now we move to concurrence calculation  $C^{A_1B_2}$  and  $C^{A_2B_1}$ . We start with  $C^{A_1B_2}$ . For this, we trace out  $A_1$  and  $B_2$ .

$$\begin{aligned} \rho^{A_1B_2} = & \langle e_{2A}, e_{1B} | \rho^{atoms} | e_{2A}, e_{1B} \rangle + \langle e_{2A}, g_{1B} | \rho^{atoms} | e_{2A}, g_{1B} \rangle \\ & + \langle g_{2A}, e_{1B} | \rho^{atoms} | g_{2A}, e_{1B} \rangle + \langle g_{2A}, g_{1B} | \rho^{atoms} | g_{2A}, g_{1B} \rangle \end{aligned} \quad (32)$$

And we find

$$\begin{aligned} z &= (x_4x_5^* + x_6x_7^* + x_6x_8^* + x_{10}x_{12}^*) \sin \alpha \cos \alpha \\ b &= (|x_2|^2 + |x_3|^2 + |x_4|^2 + |x_6|^2) \cos^2 \alpha + (|x_7|^2 + |x_{14}|^2) \sin^2 \alpha \\ c &= (|x_2|^2 + |x_3|^2 + |x_4|^2 + |x_6|^2) \cos^2 \alpha + (|x_8|^2 + |x_9|^2 + |x_{13}|^2 + |x_{15}|^2) \sin^2 \alpha \end{aligned} \quad (33)$$

Figures 6 and 7 show the plots for this concurrence for  $\alpha = \frac{\pi}{4}$  and  $\alpha = \frac{\pi}{12}$  at the first 100 seconds.

If we look at the initial state in equation (19) we see that under the transformation  $A \leftrightarrow B$  the state remain unchanged. That means  $C^{A_2B_1} = C^{A_1B_2}$  at all times.

Similarly, we can find the explicit expressions for  $C^{A_1A_2}$  and  $C^{B_1B_2}$ . The two cavities are not distinguishable, so we expect these C's to be the same and they are. The  $\rho^{A_1A_2}$  has the form

$$\rho^{A_1A_2} = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & w & z & 0 \\ 0 & z^* & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \quad (34)$$

The concurrence for this density matrix is [12]

$$C = 2 \max \{0, |z| - \sqrt{vy}\} \quad (35)$$

Where

$$\begin{aligned}
 z &= (|x_2|^2 + 2|x_4|^2 + |x_6|^2) \cos^2 \alpha + (x_7^*x_8 + x_{11}^*x_{12} + x_{12}x_{13}^* + x_{14}^*x_{15}) \sin^2 \alpha \\
 v &= (2|x_6|^2 + |x_{10}|^2) \cos^2 \alpha + (|x_3|^2 + |x_{14}|^2 + |x_{15}|^2 + |x_{16}|^2) \sin^2 \alpha \\
 y &= (|x_1|^2 + 2|x_2|^2 + |x_3|^2) \cos^2 \alpha + (|x_5|^2 + |x_7|^2 + |x_8|^2 + |x_9|^2) \sin^2 \alpha
 \end{aligned}
 \tag{36}$$

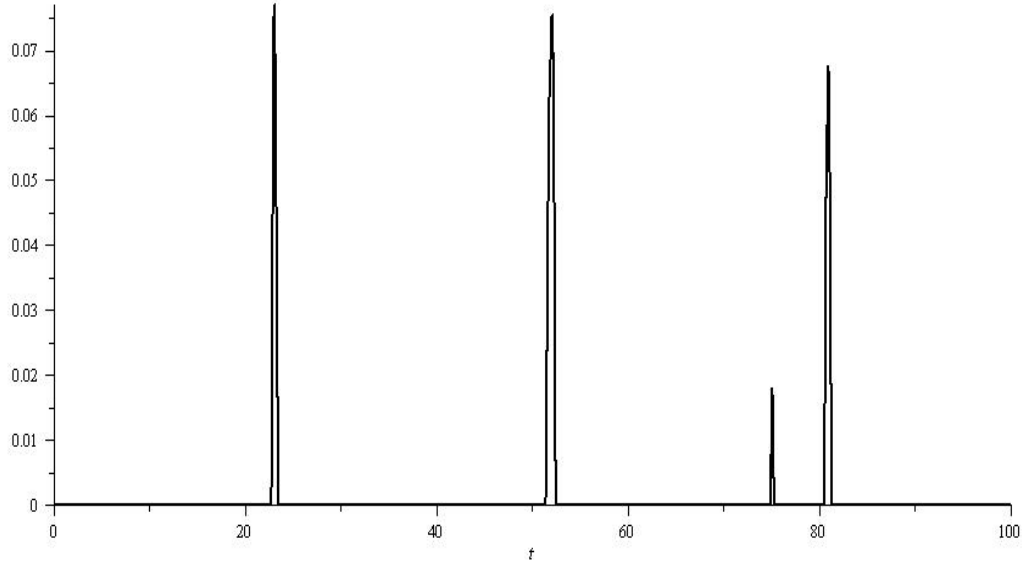


Figure 6. plot of  $A_1B_2$  concurrence for  $\alpha = \frac{\pi}{4}$

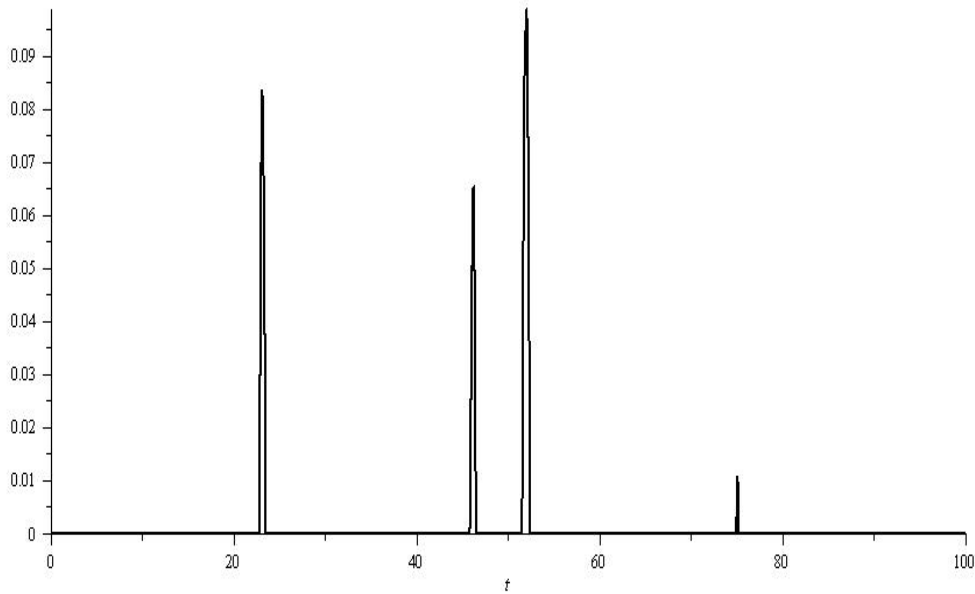


Figure 7. plot of  $A_1B_2$  concurrence for  $\alpha = \frac{\pi}{12}$

### 5. Conclusions

In this paper we considered two two-atoms Tavis-Cummings systems. These two systems have no interaction together, but one atom from each system are entangled initially ( $A_1, B_1$ ).

The entanglement between  $A_1$  and  $B_1$  for times  $t > 0$  falls to zero and remains non-entangled for a period of time till becomes entangled again. We also see that the entangled intervals are smaller than non-entangled intervals. For initially non-entangled atoms,  $A_2$  and  $B_2$ , the



entanglement appears in short intervals. But the entanglement between  $A_1$  and  $B_2$  or  $A_2$  and  $B_1$  is very negligible in very short intervals; therefore, we can claim that these atoms will not be entangled afterwards.

We can test other initial states in our problem. For example for two initially entangled atoms, we can use another initially state

$$|\Psi_{A_1 B_1}\rangle = \cos\alpha |e_{1A} g_{1B}\rangle + \sin\alpha |g_{1A} e_{1B}\rangle \quad (37)$$

and solve the problem again. Or for the non-entangled atoms, which we assumed that were in their excited states initially, we can change this prescription and solve the problem again, but the methods for this new situation are similar with the method which we used before.

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