On Fuzzy δ-Semi Connectedness

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Abstract  A. Mukherjee & S. Debnath [10] introduced the concept of δ-semi open sets in fuzzy setting and investigate their basic properties. Also S. Debnath [4] introduced the concept of fuzzy δ-semi continuous functions in fuzzy topological spaces. The purpose of the present paper is to introduce and study the concept of δ-semi connectedness in fuzzy topological spaces. It is seen that every fuzzy δ-semi separated sets is fuzzy semi separated but the converse is not true. Also every fuzzy connected set is fuzzy δ-semi connected but the converse may not be true.

Keywords  Fuzzy δ-semi open, fuzzy semi separated, fuzzy δ-semi separated, fuzzy δ-semi continuous, fuzzy semi connected, fuzzy δ-semi connected

1. Introduction and Preliminaries

According to Zadeh [17], a fuzzy subset µ in a set X ≠ {φ} is a function from X to the unit closed interval I = [0, 1] i.e., µ ∈ I+. In what follows, by (X, τ) or simply by X we mean a fuzzy topological space (fts, for short) in the sense of Chang [3]. The notations clµ, intµ and 1−µ will stand for the fuzzy closure, interior and complement of a fuzzy set µ in a fts X.

A fuzzy point x in a fts X is quasi-coincident with a fuzzy set µ denoted by x q µ if p + µ(x) > 1 [12]. The negation of this statement written as x q µ. A fuzzy set µ of a fts X is called a Q-neighbourhood (Q-nbd, for short) of a fuzzy point x if there exists a fuzzy open set λ in X such that x q λ ≤ µ [12]. A fuzzy set µ of X is called regular open if µ = int cl µ [1]. A fuzzy point x is called a δ-cluster point of a fuzzy set µ of X if every fuzzy regular open Q nbd. of x is quasi-coincident with µ. The union of all δ-cluster of µ is called δ-closure of µ and is denoted by δ cl µ. A fuzzy set µ is called δ-closed if µ = δ cl µ and the complement of such a fuzzy set is called fuzzy δ-open.

Fuzzy δ-interior of a fuzzy set µ in a fts X, denoted by δ int µ, is defined by δ int µ = 1 − δ cl (1 − µ) [11]. It is known [14] that a fuzzy set µ in a fts X is fuzzy δ-open iff µ = δ int µ, and the set of all fuzzy δ-open sets in (X, τ) forms a fuzzy topology τ δ (say) on X, called the fuzzy semi-regularization topology of (X, τ), such that τ δ ⊆ τ, and for which the fuzzy regular open sets form a fuzzy open base.

Definition 1.1: A fuzzy set λ in a fuzzy topological space X is called:

(a) fuzzy semi open [1] if λ ≤ cl (int λ)
(b) fuzzy δ-semi open [10] if ∃ a fuzzy δ-open set µ in X such that λ ≤ cl (µ).

Remark 1.2: Every fuzzy δ-semi open set is fuzzy semi open but the converse may not be true [10].

Definition 1.3: The complement of a fuzzy semi open (resp. fuzzy δ-semi open) set is called fuzzy semi closed (resp. fuzzy δ-semi closed).

Definition 1.4: The intersection of all fuzzy semi closed (resp. fuzzy δ-semi closed) sets containing a fuzzy set λ of a fuzzy topological space X is called fuzzy semi closure [16] (resp. fuzzy δ-semi closure [13]) of λ and is denoted by scl (λ) (resp. δ scl (λ)).

Definition 1.5: A mapping f : X → Y is said to be fuzzy δ-semi continuous [4] (resp. fuzzy δ-semi irresolute [4]) if f−1 (λ) is fuzzy δ-semi open in X for every fuzzy open (resp. fuzzy δ-semi open) set λ in Y.
Definition 1.6: Two nonempty fuzzy sets \( \lambda \) and \( \mu \) of a fuzzy topological space \( X \) are said to be fuzzy separated[5] (resp. fuzzy semi separated[7]) if \( cl(\lambda)q\mu \) and \( cl(\mu)q\lambda \) (resp. \( scl(\lambda)q\mu \) and \( scl(\mu)q\lambda \)).

Definition 1.7: A fuzzy set \( \eta \) in a fuzzy topological space \( X \) is said to be fuzzy connected[6] (resp. fuzzy semi connected) if \( \eta \) can not be expressed as the union of two fuzzy separated (resp. fuzzy semi separated) sets.

Remark 1.8: Every fuzzy semi connected set is fuzzy connected but the converse may not be true[7]. Throughout this paper fts \( X \) denotes a fuzzy topological space \( X \) and \( F_\delta SO(X) \) (resp. \( F_\delta SC(X) \)) denotes the family of all fuzzy \( \delta \)-semi open (resp. fuzzy \( \delta \)-semi closed) sets of \( X \).

2. Fuzzy \( \delta \)-semi Separated Sets

Definition 2.1: Two nonempty fuzzy sets \( \lambda \) and \( \mu \) in a fts \( X \) are said to be fuzzy \( \delta \)-semi separated if \( \delta - scl(\lambda)q\mu \) and \( \delta - scl(\mu)q\lambda \).

Remark 2.2: Any two fuzzy \( \delta \)-semi separated sets in a fts \( X \) are fuzzy separated but the converse is not true.

Example 2.3: Suppose \( X = \{a,b\} \) and \( \tau = \{0,1,\mu_1,\mu_2\} \), where
\[
\begin{align*}
\mu_1(a) &= 0.5, \quad \mu_1(b) = 0.3 \\
\mu_2(a) &= 0.5, \quad \mu_2(b) = 0.1
\end{align*}
\]
Then \( (X,\tau) \) is a fts. Let \( \mu_1 \) be a fuzzy set in \( X \) such that \( \mu_1(a) = 0.5, \mu_1(b) = 0.2 \). Then the fuzzy set \( \mu_1 \) and \( \mu_2 \) are fuzzy semi separated but not fuzzy \( \delta \)-semi separated in \( X \).

Theorem 2.4: Let \( \lambda \) and \( \mu \) be fuzzy \( \delta \)-semi separated sets in a fts \( X \) and \( \lambda_1 \) and \( \mu_1 \) are two nonempty fuzzy sets such that \( \lambda_1 \leq \lambda \) and \( \mu_1 \leq \mu \), then \( \lambda_1 \) and \( \mu_1 \) are fuzzy \( \delta \)-semi separated sets in \( X \).

Proof: Since \( \lambda_1 \leq \lambda \) and \( \mu_1 \leq \mu \) we have \( \delta - scl(\lambda_1) \leq \delta - scl(\lambda) \) and \( \delta - scl(\mu_1) \leq \delta - scl(\mu) \).

Therefore \( \delta - scl(\lambda)q\mu \Rightarrow \delta - scl(\lambda_1)q\mu_1 \) and \( \lambda q\delta - scl(\mu) \Rightarrow \lambda_1 q\delta - scl(\mu_1) \). Hence the theorem.

Lemma 2.5[12]: Let \( \lambda \) and \( \mu \) be two fuzzy sets in fts \( X \), then \( \lambda \leq \mu \) if and only if \( \lambda q(1-\mu) \).

Theorem 2.6: Let \( \lambda,\mu \in F_\delta SO(X) \). Then \( \lambda \) and \( \mu \) are fuzzy \( \delta \)-semi separated in \( X \) if and only if \( \lambda q\mu \).

Proof: Necessary: If \( \lambda q\mu \) then there exists a point \( x \in X \) such that \( \lambda(x) + \mu(x) > 1 \). This implies that \( \delta - scl(\lambda(x)) + \mu(x) > 1 \) and \( \lambda(x) + \delta - scl(\mu(x)) > 1 \). Hence \( \delta - scl(\lambda)q\mu \) and \( \lambda q\delta - scl(\mu) \), which is contradiction. As \( \lambda \) and \( \mu \) are fuzzy \( \delta \)-semi separated sets in \( X \) hence \( \lambda q\mu \).

Sufficient: Suppose that \( \lambda q\mu \), then \( \lambda \leq (1-\mu) \).

Therefore, \( \delta - scl(\lambda) \leq \delta - scl(1-\mu) \), because \( 1-\mu \in F_\delta SC(X) \).

Hence by lemma 2.5 \( \delta - scl(\lambda)q\mu \). Similarly, \( \lambda q\delta - scl(\lambda)q\mu \).

Hence the theorem.

Theorem 2.7: Let \( \lambda,\mu \in F_\delta SC(X) \). Then \( \lambda \) and \( \mu \) are fuzzy \( \delta \)-semi separated in \( X \) if and only if \( \lambda q\mu \).

Proof: Obvious.

Theorem 2.8: Let \( \lambda,\mu \in F_\delta SO(X) \). Then the fuzzy sets \( A_\lambda \mu = \lambda \cap (1-\mu) \) and \( A_\mu \lambda = \mu \cap (1-\lambda) \) are fuzzy \( \delta \)-semi separated in \( X \).

Proof: Since \( A_\lambda \mu \leq (1-\mu) \), so \( \delta - scl(A_\lambda \mu) \leq \delta - scl(1-\mu) = (1-\mu) \). because \( \mu \in F_\delta SO(X) \).

And so by lemma 2.5, \( \delta - scl(A_\lambda \mu)q\mu \). Thus, \( \delta - scl(A_\lambda \mu)q(A_\mu \lambda) \). Similarly, \( \delta - scl(A_\mu \lambda)q(A_\lambda \mu) \). Hence \( A_\lambda \mu \) and \( A_\mu \lambda \) are fuzzy \( \delta \)-semi separated sets.

Theorem 2.9: Let \( \lambda,\mu \in F_\delta SC(X) \). Then the fuzzy sets \( A_\lambda \mu = \lambda \cap (1-\mu) \) and \( A_\mu \lambda = \mu \cap (1-\lambda) \) are fuzzy \( \delta \)-semi separated in \( X \).

Proof: Obvious.
\textbf{Theorem 2.10}: Two fuzzy sets $\lambda$ and $\mu$ of a fts $X$ are fuzzy $\delta$-semi separated if and only if there exists fuzzy sets $\alpha, \beta \in F_\delta \text{SO}(X)$ such that $\lambda \leq \alpha, \mu \leq \beta, \lambda \q \beta$ and $\mu \alpha$.

\textbf{Proof}: Necessary: Let $\lambda$ and $\mu$ be two fuzzy $\delta$-semi separated sets in $X$. Put $\beta = 1 - (\delta - \text{scl}(\lambda))$ and $\alpha = 1 - (\delta - \text{scl}(\mu))$, then $\alpha, \beta \in F_\delta \text{SO}(X)$ such that $\lambda \leq \alpha, \mu \leq \beta, \lambda \q \beta$ and $\mu \alpha$.

Sufficient: Let $\alpha, \beta \in F_\delta \text{SO}(X)$ such that $\lambda \leq \alpha, \mu \leq \beta, \lambda \q \beta$ and $\mu \alpha$.

Now $(1 - \beta), (1 - \alpha) \in F_\delta \text{SC}(X)$, we have $(\delta - \text{scl}(\lambda)) \leq 1 - \beta \leq 1 - \mu$ and $(\delta - \text{scl}(\mu)) \leq 1 - \alpha \leq 1 - \lambda$. Therefore by lemma 2.5, $(\delta - \text{scl}(\lambda)) \q \mu$ and $(\delta - \text{scl}(\mu)) \q \lambda$. Hence $\lambda$ and $\mu$ are fuzzy $\delta$-semi separated in $X$.

3. Fuzzy $\delta$-semi Connectedness

\textbf{Definition 3.1}: A fuzzy set $\alpha$ in a fts $X$ is said to be fuzzy $\delta$-semi connected if it cannot be expressed as the union of two fuzzy $\delta$-semi separated sets.

\textbf{Remark 3.2}: Every fuzzy connected set is fuzzy $\delta$-semi connected but the converse may not be true. For the fuzzy set $\mu$, considered in example 2.3 is fuzzy $\delta$-semi connected but not fuzzy semi connected.

\textbf{Theorem 3.3}: Let $\lambda$ and $\mu$ be two fuzzy $\delta$-semi separated sets in a fts $X$ and $\alpha$ be a fuzzy $\delta$-semi connected set in $X$ such that $\alpha \leq \lambda \cup \mu$. Then exactly one of the following conditions (a) and (b) holds:

(a) $\alpha \leq \lambda$ and $\alpha \cap \mu = 0$

(b) $\alpha \leq \mu$ and $\alpha \cap \lambda = 0$

\textbf{Proof}: We first note that when $\alpha \cap \mu = 0$, then $\alpha \leq \lambda$, since $\alpha \leq \lambda \cup \mu$. Similarly, since $\alpha \cap \lambda = 0$, we have $\alpha \leq \mu$. Now, since $\alpha \leq \lambda \cup \mu$, both $\alpha \cap \lambda = 0$ and $\alpha \cap \mu = 0$ cannot hold simultaneously.

Again if $\alpha \cap \mu \neq 0$ and $\alpha \cap \lambda \neq 0$, then $\alpha \cap \lambda$ and $\alpha \cap \mu$ are fuzzy $\delta$-semi separated in $X$ such that $\alpha = (\alpha \cap \lambda) \cup (\alpha \cap \mu)$, contradicting the fuzzy $\delta$-semi connectedness of $\alpha$. Hence exactly one of the conditions (a) and (b) must holds.

\textbf{Theorem 3.4}: Let $\alpha$ and $\beta$ be two fuzzy sets of a fts $X$. If $\alpha$ is fuzzy $\delta$-semi connected and $\alpha \leq \beta \leq \delta - \text{scl}(\alpha)$, then $\beta$ is fuzzy $\delta$-semi connected.

\textbf{Proof}: If $\alpha = 0$ then the result is true. Let $\alpha \neq 0$, suppose $\beta$ is not fuzzy $\delta$-semi connected then $\exists$ two fuzzy $\delta$-semi separated sets $\lambda$ and $\mu$ in $X$ such that $\beta = \lambda \cup \mu$. Since $\alpha$ is fuzzy $\delta$-semi connected and $\alpha \leq \beta \leq \lambda \cup \mu$, by theorem 3.3 $\alpha \leq \lambda$ and $\alpha \cap \mu = 0$ or $\alpha \leq \mu$ and $\alpha \cap \lambda = 0$. Let $\alpha \leq \lambda$ and $\alpha \cap \mu = 0$, then $\mu = \mu \cap \beta \leq \mu \cap \delta - \text{scl}(\alpha)$.

It follows that $\mu = \mu \cap \delta - \text{scl}(\lambda) \leq \mu \cap (1 - \mu) \leq \mu$. Theorem 3.6: Let $f : X \rightarrow Y$ be a fuzzy $\delta$-semi irresolute surjective mapping. If $\eta$ is a fuzzy $\delta$-semi connected subset in $X$, then $f(\eta)$ is fuzzy $\delta$-semi connected subset in $Y$.

\textbf{Proof}: Suppose that $f(\eta)$ is not fuzzy $\delta$-semi connected in $Y$. Then there exists fuzzy $\delta$-semi separated subset $\lambda$ and $\mu$ in $Y$ such that $f(\eta) = \lambda \cup \mu$. By theorem 2.10, there exists fuzzy $\delta$-semi open subsets $\alpha$ and $\beta$ such that $\lambda \leq \alpha, \mu \leq \beta, \lambda \q \beta$ and $\mu \alpha$. Since $f$ is fuzzy $\delta$-semi irresolute,

$\eta = f^{-1}(f(\eta)) = f^{-1}(\lambda) \cup f^{-1}(\mu)$

$\leq f^{-1}(\lambda \cup \mu)$.

Also it can be easily seen that $f^{-1}(\lambda) \leq f^{-1}(\alpha)$, $f^{-1}(\mu) \leq f^{-1}(\beta)$, $f^{-1}(\lambda) \q f^{-1}(\beta)$ and $f^{-1}(\mu) \q f^{-1}(\alpha)$. Thus $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are fuzzy $\delta$-semi separated in $X$. Hence $\eta$ is not fuzzy $\delta$-semi connected, which is a contradiction.

\textbf{Theorem 3.5}: Let $f : X \rightarrow Y$ be a fuzzy $\delta$-semi continuous surjective mapping. If $\eta$ is a fuzzy $\delta$-semi connected subset in $X$, then $f(\eta)$ is fuzzy connected subset in $Y$.

\textbf{Proof}: Analogues to that theorem above.
4. Conclusions

Based on $\delta$-semi open sets and $\delta$-Semi continuous functions, $\delta$-semi connectedness was introduced, which is the weaker form of different existing connectedness in fuzzy topological spaces.

REFERENCES


