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Averaging of Fuzzy Integrodifferential Inclusions

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Abstract In this article we prove the substantiation of the method of averaging for the fuzzy integrodifferential inclusions with small parameter. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

Keywords Fuzzy Integrodifferential Inclusion, Fuzzy Systems, Method of Averaging

1. Introduction

Many important problems of analytical dynamics are described by the nonlinear mathematical models that as a rule are presented by the nonlinear differential or the integrodifferential equations. The absence of exact universal research methods for nonlinear systems has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

The averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. Averaging theory for ordinary differential equations has a rich history, dating to back to the work of N.M. Krylov and N.N. Bogoliubov[1], and has been used extensively in engineering applications[2-6]. Books that cover averaging theory for differential equations and inclusions include[7-10].

In recent years, the fuzzy set theory introduced by Zadeh[11] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of science as physical, mathematical, differential equations and engineering sciences. Recently there have been new advances in the theory of fuzzy differential equations[12-16], fuzzy integrodifferential equations[17-20], differential inclusions with fuzzy right-hand side[21-24] and fuzzy differential inclusions[25-27] as well as in the theory of control fuzzy differential equations[28-30], control fuzzy integrodifferential equations[31-35], controlfuzzy differential inclusions[36-39], and control fuzzy integrodifferential inclusions[40]. In works[41-48] various schemes of an average for the fuzzy differential equations and inclusions have been considered. In this article we prove the substantiation of the method of full averaging for the integrodifferential inclusions with small parameter on the metric space $(E^n;D)$. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

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2. Preliminaries

Let $comp(R^n)\left(conv(R^n)\right)$ be a set of all nonempty (convex) compact subsets from the space R^n,

$$h(A,B)=\min_{r\ge 0}\left\{S_r(A)\supset B, S_r(B)\supset A\right\}$$

be Hausdorff distance between sets A and B, $S_r(A)$ is r-neighborhood of set A.

Let E^n be the set of all $u:R^n\to[0,1]$ such that u satisfies the following conditions:

- u is normal, that is, there exists an $x_0\in R^n$ such that $u(x_0)=1$;
- u is fuzzy convex, that is,

$$u(\lambda x+(1-\lambda)y)\ge\min\{u(x),u(y)\}$$

- for any $x,y\in R^n$ and $0\le\lambda\le 1$;
- u is upper semicontinuous,
- $[u]^0=cl\left\{x\in R^n: u(x)>0\right\}$ is compact.

If $u\in E^n$, then u is called a fuzzy number, and E^n is said to be a fuzzy number space. For $0<\alpha\le 1$, denote

$$[u]^\alpha=\left\{x\in R^n: u(x)\ge\alpha\right\}.$$

Then from 1)-4), it follows that the α-level set $[u]^\alpha\in conv(R^n)$ for all $0\le\alpha\le 1$.

Let $\hat{\theta}$ be the fuzzy mapping defined by $\hat{\theta}(x)=0$ if $x\ne 0$ and $\hat{\theta}(x)=1$.

Define $D:E^n\times E^n\to[0,\infty)$ by the relation

$$D(u,v)=\sup_{0\le\alpha\le 1} h\left([u]^\alpha,[v]^\alpha\right),$$

where h is the Hausdorff metric defined in $comp(R^n)$.

Then D is a metric in E^n.

Further we know that[49]:

1. (E^n, D) is a complete metric space,
2. $D(u+w, v+w) = D(u,v)$ for all $u,v,w \in E^n$,
3. $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u,v \in E^n$ and $\lambda \in R$.

Definition 1.[15] A mapping $F:[0,T] \to E^n$ is measurable if for all $\alpha \in [0,1]$ the set-valued map $F_\alpha : [0,T] \to conv(R^n)$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.

Definition 2.[15] A mapping $F:[0,T] \to E^n$ is said to be integrably bounded if there is an integrable function $h(t)$ such that $\|x(t)\| \le h(t)$ for every $x(t) \in F_0(t)$.

Definition 3.[15] The integral of a fuzzy mapping $F:[0,T] \to E^n$ is defined levelwise by $\left[\int_0^T F(t)dt\right]^\alpha = \int_0^T F_\alpha(t)dt$. The set $\int_0^T F_\alpha(t)dt$ of all $\int_0^T f(t)dt$ such that $f:[0,T] \to R^n$ is a measurable selection for $F_\alpha : [0,T] \to conv(R^n)$ for all $\alpha \in [0,1]$.

Definition 4.[15] A measurable and integrably bounded mapping $F:[0,T] \to E^n$ is said to be integrable over $[0,T]$ if $\int_0^T F(t)dt \in E^n$.

Note that if $F:[0,T] \to E^n$ is measurable and integrably bounded, then F is integrable. Further if $F:[0,T] \to E^n$ is continuous, then it is integrable.

Now we consider following fuzzy integrodifferential inclusion

$$\dot{x} \in F(t,x) + \int_o^t \Phi(t,s,x(s))ds \quad x(0) = x_0, \tag{1}$$

where $\dot{x}$ means dx/dt; $x \in R^n$ is the state;

$F: R_+ \times R^n \to E^n$

is a fuzzy mapping, $\Phi: R_+ \times R_+ \times R^n \to E^n$ is a fuzzy mapping, $t,s \in R_+$, $x_0 \in R^n$.

We interpret[21-24] the fuzzy differential inclusion (1) as a family of integrodifferential inclusions

$$\dot{x}_\alpha \in [F(t,x_\alpha)]^\alpha + \int_o^t [\Phi(t,s,x_\alpha(s))]^\alpha ds \quad x_\alpha(0) = x_0, \tag{2}$$

where the subscript α indicates that the α-level set of a fuzzy set is involved (the system (2) can only have any significance as a replacement for (1) if the solutions generate fuzzy sets (fuzzy R-solution)[24]).

Let $X(t)$ denotes the fuzzy R-solution of the fuzzy integrodifferential inclusion (1).

3. The Scheme of an Average

In this section we consider the Cauchy problem with small parameter

$$\dot{x} \in \varepsilon\left[F(t,x) + \int_0^t \Phi(t,s,x(s))ds\right], \quad x(0) = x_0, \tag{3}$$

where $\varepsilon > 0$ is a small parameter.

In this section we associate with the equation (3) the following averaged integrodifferential equation

$$\dot{y} \in \varepsilon\left[\bar{F}(y) + \int_0^t \bar{\Phi}(t,y(s))ds\right], \quad y(0) = x_0, \tag{4}$$

where

$$\lim_{T\to\infty} D\left(\frac{1}{T}\int_0^T F(t,x)dt, \bar{F}(x)\right) = 0, \tag{5}$$

$$\bar{\Phi}(t,x) = \begin{cases} \frac{1}{t}\Phi_1(t,x), & t > 0, \\ \lim\limits_{t\to 0+} \frac{1}{t}\Phi_1(t,x), & t = 0, \end{cases} \quad \Phi_1(t,x) = \int_0^t \Phi(t,s,x)ds. \tag{6}$$

Remark. In this paper we will consider a case when the limits (5), (6) exist.

Theorem. Let in domain

$$Q = \{(t,x) \mid t \ge 0,\ x \in G \subset R^n\}$$

the following hold:

1) $F(t,x)$ is continuous in $(t,x) \in R_+ \times G$;

2) $\Phi(t,s,x)$ is continuous in $(t,s,x) \in R_+ \times R_+ \times G$;

3) there exist constants λ, M_1 such that

$$D(F(t,x), \hat{\theta}) \le M_1, \quad D(F(t,x_1), F(t,x_2)) \le \lambda \|x_1 - x_2\|,$$

for all $(t,x) \in Q$, and any $x_1, x_2 \in G$;

4) there exist continuous functions $\mu(t,s)$, $K(t,s)$, and constant M_2 such that

$$D(\Phi(t,s,x_1), \Phi(t,s,x_2)) \le \mu(t,s)\|x_1 - x_2\|,$$

$$D(\Phi(t,s,x), \hat{\theta}) \le K(t,s),$$

$$\int_{t_1}^{t_2}\int_0^t K(t,s)dsdt \le M_2(t_2 - t_1),$$

for any $0 \le t_1 \le t_2 < \infty$, and any $x_1, x_2 \in G$;

5) there exist constants M_3, and M_4 such that

$$\int_{t_1}^{t_2}\int_0^t \mu(t,s)dsdt \le M_3(t_2 - t_1),$$

$$\int_{t_1}^{t_2}\int_0^t \mu(t,s)(t-s)dsdt \le M_4(t_2 - t_1),$$

for any $0 \le t_1 \le t_2 < \infty$;

6) there exist continuous functions $\upsilon(t)$, $\bar{K}(t)$, and constants υ_1, υ_2, K_1 such that

$$D(\bar{\Phi}(t,x), \hat{\theta}) \le \bar{K}(t), \quad D(\bar{\Phi}(t,x_1), \bar{\Phi}(t,x_2)) \le \upsilon(t)\|x_2 - x_1\|,$$

$$\int_{t_1}^{t_2} t\bar{K}(t)dt \le K_1(t_2 - t_1), \quad \int_{t_1}^{t_2} t\upsilon(t)dt \le \upsilon_1(t_2 - t_1),$$

$$\int_{t_1}^{t_2} t^2\upsilon(t)dt \le \upsilon_2(t_2 - t_1),$$

for any $0 \le t_1 \le t_2 < \infty$ and $x_1, x_2 \in G$;

7) the limits (5),(6) exist uniformly in $x \in G$;

8) for any $x_0 \in G' \subset G$ and $t \ge 0$ the fuzzy R-solution of the system (4) together with a σ-neighbourhood belong to the domain G.

Then for any $\eta > 0$ and $L > 0$ there exists $\varepsilon^0(\eta, L) \in (0, \sigma]$ such that for all $\varepsilon \in (0, \varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statement fulfil:

$$D\big(X(t), Y(t)\big) < \eta, \tag{7}$$

where $X(t)$, $Y(t)$ are the fuzzy R-solutions of the initial and the full averaged equations.

Proof. Let $\eta > 0$ is any constant. In the beginning, us prove the validity of the inclusion for all $t \in [0, L\varepsilon^{-1}]$

$$X(t) \subset Y(t) + S_\eta(\hat{\theta}), \tag{8}$$

where $\left[S_\eta(\hat{\theta})\right]^\alpha \equiv S_\eta(0)$ for all $\alpha \in [0,1]$. Let $\beta \in [0,1]$ is arbitrary and we prove the validity of the inclusion

$$[X(t)]^\beta \subset [Y(t)]^\beta + S_\eta(0)$$

for all $t \in [0, L\varepsilon^{-1}]$.

We consider the integrodifferential inclusion

$$\dot{x}_\beta \in \varepsilon\left[F(t, x_\beta)\right]^\beta + \varepsilon\left[\int_0^t \Phi\big(t, s, x_\beta(s)\big)ds\right]^\beta, \ x_\beta(0) = x_0. \tag{9}$$

Let $x_\beta(t)$ is any solution of the system (9). Then

$$x_\beta(t) = x_0 + \varepsilon\int_0^t v(\tau)d\tau, \tag{10}$$

where $v(t) \in \left[F\big(t, x_\beta(t)\big)\right]^\beta + \left[\int_0^t \Phi\big(t, s, x_\beta(s)\big)ds\right]^\beta$.

Divide the interval $[0, L\varepsilon^{-1}]$ into partial intervals by the points $t_i = iL/m\varepsilon$, $i = 0, \dots, m$, $m \in \mathrm{N}$. We consider the function

$$x_\beta^1(t) = x_\beta^1(t_i) + \varepsilon\int_{t_i}^t v^1(\tau)d\tau, \ x_\beta^1(0) = x_0, \ t \in [t_i, t_{i+1}], \tag{11}$$

where

$$\left\|v^1(t) - v(t)\right\| = \min_{w(t) \in \left[F\left(t, x_\beta^1(t_i)\right)\right]^\beta + \left[\int_0^t \Phi\left(t, s, x_\beta^1(t_i)\right)ds\right]^\beta} \left\|w(t) - v(t)\right\|.$$

By conditions 3), and 4) of the theorem, we have

$$\left\|x_\beta(t) - x_\beta(t_i)\right\| \le \varepsilon\int_0^t \left(M_1 + \int_0^\tau K(\tau, s)ds\right)d\tau \le$$

$$\le \varepsilon(M_1 + M_2)(t - t_i) \le (M_1 + M_2)\frac{L}{m},$$

$$\left\|x_\beta(t) - x_\beta(s)\right\| \le \varepsilon(M_1 + M_2)(t - s),$$

for all $t \in [t_i, t_{i+1}]$ and $s \le t$.

Also,

$$\left\|x_\beta^1(t) - x_\beta^1(t_i)\right\| \le (M_1 + M_2)\frac{L}{m}.$$

Let $\delta_i = \left\|x_\beta(t_i) - x_\beta^1(t_i)\right\|$. Then, we get

$$\left\|x_\beta(t) - x_\beta^1(t_i)\right\| \le \left\|x_\beta(t) - x_\beta(t_i)\right\| + \left\|x_\beta(t_i) - x_\beta^1(t_i)\right\| \le$$

$$\le \delta_i + \varepsilon(M_1 + M_2)(t - t_i). \tag{12}$$

By (12), and conditions 3), 5) of the theorem, we obtain

$$h\Bigg(\left[F\big(t, x_\beta(t)\big)\right]^\beta + \left[\int_0^t \Phi\big(t, s, x_\beta(s)\big)ds\right]^\beta,$$

$$\left[F\big(t, x_\beta^1(t_i)\big)\right]^\beta + \left[\int \Phi\big(t, s, x_\beta^1(t_i)\big)ds\right]^\beta\Bigg) \le$$

$$\le \left\|x_\beta(t) - x_\beta^1(t_i)\right\| + \int_0^t \mu(t, s)\left\|x_\beta(s) - x_\beta^1(t_i)\right\|ds \le$$

$$\le \left[\delta_i + \varepsilon(M_1 + M_2)(t - t_i)\right]\left(\lambda + \int_0^t \mu(t, s)ds\right) +$$

$$+\varepsilon(M_1 + M_2)\int_0^t \mu(t, s)(t - s)ds = a(\varepsilon, t, t_i),$$

for all $t \in [t_i, t_{i+1}]$.

It follows, we have $\left\|v(t) - v^1(t)\right\| \le a(\varepsilon, t, t_i)$, for all $t \in [t_i, t_{i+1}]$.

As

$$\left\|x_\beta(t) - x_\beta^1(t)\right\| \le \left\|x_\beta(t_i) - x_\beta^1(t_i)\right\| + \varepsilon\int_0^t \left\|v(\tau) - v^1(\tau)\right\|d\tau,$$

for all $t \in [t_i, t_{i+1}]$, then we get

$$\delta_{i+1} \le \delta_i\left(1 + \frac{L(\lambda + M_3)}{m}\right) + a_1(\varepsilon, m),$$

where

$$a_1(\varepsilon, m) = (M_1 + M_2)\frac{L^2}{m^2}\left(\frac{\lambda}{2} + M_3\right) + \varepsilon\frac{M_4 L(M_1 + M_2)}{m}.$$

Therefore

$$\delta_{i+1} \le \frac{M_1 + M_2}{\lambda + M_3}\left(\frac{L}{2m}(\lambda + 2M_3) + \varepsilon M_4\right)\left(e^{L(\lambda + M_3)} - 1\right),$$

for all $i = \overline{0, m-1}$.

Then we obtain

$$\left\|x_\beta(t)-x_\beta^1(t)\right\| \le 2(M_1+M_2)\frac{L}{m}+ + \frac{M_1+M_2}{\lambda+M_3}\left(\frac{L}{2m}(\lambda+2M_3)+\varepsilon M_4\right)\left(e^{L(\lambda+M_3)}-1\right).$$

Now we have function $\xi_\beta(t)$ such that

$$\xi_\beta(t)=\xi_\beta(t_i)+\varepsilon\int_0^t g(\tau)d\tau,\quad \xi_\beta(0)=x_0,\quad t\in[t_i,t_{i+1}],$$

where

$$\left\|v^1(t)-g(t)\right\| = \min_{w(t)\in\left[\bar F\left(x_\beta^1(t_i)\right)\right]^\beta+\left[t\Phi\left(t,x_\beta^1(t_i)\right)\right]^\beta}\left\|v^1(t)-w(t)\right\|.$$

By condition 7) of the theorem, for any $\eta_1>0$, there exists $\varepsilon^0(\eta_1,L)>0$ such that the following estimate is true for $0<\varepsilon<\varepsilon^0$:

$$h\left(\frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}}\left(\left[F\left(\tau,x_\beta^1(t_i)\right)\right]^\beta+\int_0^t\left[\Phi\left(\tau,s,x_\beta^1(t_i)\right)\right]^\beta ds\right)d\tau,\right.$$

$$\left.\left[\bar F\left(x_\beta^1(t_i)\right)\right]^\beta+\frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}}\tau\left[\bar\Phi\left(\tau,x_\beta^1(t_i)\right)\right]^\beta d\tau\right)\le\eta_1.$$

Hence,

$$\frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}}\left\|v^1(\tau)-g(\tau)\right\|d\tau\le\eta_1.$$

Then

$$\left\|x_\beta^1(t_{i+1})-\xi_\beta(t_{i+1})\right\|\le L\eta_1.$$

Also we have

$$\left\|\xi_\beta(t)-\xi_\beta(t_i)\right\|\le\frac{L}{m}(M_1-M_2),$$

$$\left\|x_\beta^1(t)-\xi_\beta(t)\right\|\le\frac{2L}{m}(M_1+M_2)+L\eta_1,$$

for all $t\in[t_i,t_{i+1}]$.

From here and by conditions 3), 6) of the theorem, we obtain

$$h\left(\left[\bar F\left(\xi_\beta(t)\right)\right]^\beta+\int_0^t\left[\bar\Phi\left(t,\xi_\beta(s)\right)\right]^\beta ds,\right.$$

$$\left.\left[\bar F\left(x_\beta^1(t_i)\right)\right]^\beta+t\left[\bar\Phi\left(t,x_\beta^1(t_i)\right)\right]^\beta\right)\le b(m,\varepsilon,t,\eta_1),$$

where

$$b(m,\varepsilon,t,\eta_1)=(\lambda+t\upsilon(t))\left((M_1+M_2)\frac{L}{m}+L\eta_1\right)+ + \frac{\varepsilon t^2\upsilon(t)}{2}(M_1+M_2).$$

Therefore,

$$\rho\left(\dot\xi_\beta(t),\varepsilon\left[\bar F\left(\xi_\beta(t)\right)\right]^\beta+\varepsilon\int_0^t\left[\bar\Phi\left(t,\xi_\beta(s)\right)\right]^\beta ds\right)\le$$

$$\le h\left(\varepsilon\left[\bar F\left(x_\beta^1(t_i)\right)\right]^\beta+\varepsilon t\left[\bar\Phi\left(t,x_\beta^1(t_i)\right)\right]^\beta,\right.$$

$$\left.\varepsilon\left[\bar F\left(\xi_\beta(t)\right)\right]^\beta+\varepsilon\int_0^t\left[\bar\Phi\left(t,\xi_\beta(s)\right)\right]^\beta ds\right)\le$$

$$\le\varepsilon b(m,\varepsilon,t,\eta_1). \tag{13}$$

From (13) it follows the existence of such a solution $y_\beta(t)$ of inclusion

$$\dot y_\beta\in\varepsilon\left[\bar F(y)\right]^\beta+\varepsilon\int_0^t\left[\bar\Phi\left(t,y_\beta(s)\right)\right]^\beta ds,\quad y_\beta(0)=x_0,$$

that for all $t\in[0,L\varepsilon^{-1}]$

$$\left\|\xi_\beta(t)-y_\beta(t)\right\|\le Le^{(\lambda+\upsilon_1)L}c(m,\varepsilon,\eta_1),$$

where

$$c(m,\varepsilon,\eta_1)=(\lambda+\upsilon_1)L\left(\frac{M_1+M_2}{m}+\eta_1\right)+\frac{\varepsilon(M_1+M_2)\upsilon_2}{2}.$$

Than we obtain

$$\left\|x_\beta(t)-y_\beta(t)\right\|\le 4(M_1+M_2)\frac{L}{m}+L\eta_1+a_1(m,\varepsilon)\left(e^{L(\lambda+M_3)}-1\right)+ + e^{(\lambda+\upsilon_1)L}Lc(m,\varepsilon,\eta_1).$$

Also $\lim\limits_{\substack{\varepsilon\to0\\ m\to\infty}}a_2(m,\varepsilon)=0$, and $\lim\limits_{\substack{\varepsilon\to0\\ m\to\infty}}c(m,\varepsilon,\eta_1)=0$, then there exist m, ε, and η_1 such that $\left\|x_\beta(t)-y_\beta(t)\right\|<\eta$, for all $t\in[0,L\varepsilon^{-1}]$. Hence $\left[X(t)\right]^\beta\subset\left[Y(t)\right]^\beta+S_\eta(0)$, for all $t\in[0,L\varepsilon^{-1}]$. as $\beta\in[0,1]$ is arbitrary, then $X(t)\subset Y(t)+S_\eta(\hat\theta)$, for all $t\in[0,L\varepsilon^{-1}]$.

Now, we proof, that $Y(t)\subset X(t)+S_\eta(\hat\theta)$, for all $t\in[0,L\varepsilon^{-1}]$. Also, let $\beta\in[0,1]$ is arbitrary and we prove the validity of the inclusion $\left[Y(t)\right]^\beta\subset\left[X(t)\right]^\beta+S_\eta(0)$ for all $t\in[0,L\varepsilon^{-1}]$.

We consider the integrodifferential inclusion

$$\dot y_\beta\in\varepsilon\left[\bar F(y_\beta)\right]^\beta+\varepsilon\left[\int_0^t\bar\Phi\left(t,y_\beta(s)\right)ds\right]^\beta,\quad y_\beta(0)=x_0. \tag{14}$$

Let $y_\beta(t)$ is any solution of the system (14). Then

$$y_\beta(t)=x_0+\varepsilon\int_0^t z(\tau)d\tau, \tag{15}$$

where

$$z(t) \in \left[\bar{F}\left(y_\beta(t)\right)\right]^\beta + \left[\int_0^t \bar{\Phi}\left(t, y_\beta(s)\right) ds\right]^\beta .$$

Divide the interval $[0, L\varepsilon^{-1}]$ into partial intervals by the points $t_i = iL/m\varepsilon$, $i = 0,...,m$, $m \in \mathrm{N}$. We consider the function

$$y_\beta^1(t) = y_\beta^1(t_i) + \varepsilon \int_{t_i}^t z^1(\tau) d\tau, \ y_\beta^1(0) = x_0, \ t \in [t_i, t_{i+1}],$$

where

$$\left\|z^1(t) - z(t)\right\| = \min_{w(t) \in \left[\bar{F}\left(y_\beta^1(t_i)\right)\right]^\beta + \left[\int_0^t \bar{\Phi}\left(t, y_\beta^1(t_i)\right) ds\right]^\beta} \left\|w(t) - z(t)\right\| .$$

By conditions 6), and 8) of the theorem, we have

$$\left\|y_\beta(t) - y_\beta(t_i)\right\| \le (M_1 + M_2)\frac{L}{m},$$

$$\left\|y_\beta(t) - y_\beta(s)\right\| \le \varepsilon (M_1 + M_2)(t - s),$$

$$\left\|y_\beta^1(t) - y_\beta^1(t_i)\right\| \le (M_1 + M_2)\frac{L}{m},$$

for all $t \in [t_i, t_{i+1}]$ and $s \le t$.

Let $\bar{\delta}_i = \left\|y_\beta(t_i) - y_\beta^1(t_i)\right\|$. Then (similarly (12)), we get

$$\left\|y_\beta(t) - y_\beta^1(t_i)\right\| \le \bar{\delta}_i + \varepsilon (M_1 + M_2)(t - t_i),$$

and

$$\left\|z(t) - z^1(t)\right\| \le \left[\bar{\delta}_i + \varepsilon (M_1 + M_2)(t - t_i)\right]\left(\lambda + t\upsilon(t)\right) + \\ + \varepsilon (M_1 + M_2)\frac{t^2}{2}\upsilon(t).$$

As

$$\left\|y_\beta(t) - y_\beta^1(t)\right\| \le \left\|y_\beta(t_i) - y_\beta^1(t_i)\right\| + \varepsilon \int_0^t \left\|z(\tau) - z^1(\tau)\right\| d\tau,$$

for all $t \in [t_i, t_{i+1}]$, then we get

$$\bar{\delta}_{i+1} \le \bar{\delta}_i \left(1 + \frac{L(\lambda + \upsilon_1)}{m}\right) + \bar{a}_1(\varepsilon, m),$$

where

$$\bar{a}_1(\varepsilon, m) = (M_1 + M_2)\frac{L^2}{m^2}(\lambda + \upsilon_1) + \varepsilon (M_1 + M_2)\frac{\upsilon_2 L}{2m}.$$

Therefore for all $i = \overline{0, m-1}$

$$\bar{\delta}_{i+1} \le \left(e^{L(\upsilon_1 + \lambda)} - 1\right)\bar{a}_2(m, \varepsilon),$$

where

$$\bar{a}_2(m, \varepsilon) = (M_1 + M_2)\left(\frac{L}{m} + \varepsilon \frac{\upsilon_2}{2(\lambda + \upsilon_1)}\right).$$

Then, we obtain

$$\left\|y_\beta(t) - y_\beta^1(t)\right\| \le 2(M_1 + M_2)\frac{L}{m} + \left(e^{L(\upsilon_1 + \lambda)} - 1\right)\bar{a}_2(m, \varepsilon).$$

Now we have function $\zeta_\beta(t)$ such that

$$\zeta_\beta(t) = \zeta_\beta(t_i) + \varepsilon \int_0^t r(\tau) d\tau, \ \zeta_\beta(0) = x_0, \ t \in [t_i, t_{i+1}],$$

where

$$\left\|z^1(t) - r(t)\right\| = \min_{w(t) \in \left[F\left(t, y_\beta^1(t_i)\right)\right]^\beta + \int_0^t \left[\Phi\left(t, s, y_\beta^1(t_i)\right)\right]^\beta ds} \left\|z^1(t) - w(t)\right\|.$$

By condition 7) of the theorem, for any $\eta_2 > 0$, there exists $\varepsilon^1(\eta_2, L) > 0$ such that the following estimate is true for $0 < \varepsilon < \varepsilon^1$:

$$h\left(\frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}}\left(\left[F\left(\tau, y_\beta^1(t_i)\right)\right]^\beta + \int_0^t \left[\Phi\left(\tau, s, y_\beta^1(t_i)\right)\right]^\beta ds\right) d\tau, \right. \\ \left. \left[\bar{F}\left(y_\beta^1(t_i)\right)\right]^\beta + \frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}} \tau\bar{\Phi}\left(\tau, y_\beta^1(t_i)\right) d\tau\right) \le \eta_2 .$$

Hence,

$$\frac{\varepsilon m}{L}\int_{t_i}^{t_{i+1}}\left\|z^1(\tau) - r(\tau)\right\| d\tau \le \eta_2 .$$

Then

$$\left\|y_\beta^1(t_{i+1}) - \zeta_\beta(t_{i+1})\right\| \le L\eta_2 .$$

Also we have

$$\left\|\zeta_\beta(t) - \zeta_\beta(t_i)\right\| \le \frac{L}{m}(M_1 - M_2),$$

$$\left\|y_\beta^1(t) - \zeta_\beta(t)\right\| \le \frac{2L}{m}(M_1 + M_2) + L\eta_2,$$

for all $t \in [t_i, t_{i+1}]$.

From here and by conditions 3), and 6) of the theorem, we obtain

$$h\left(\left[F\left(t, \zeta_\beta(t)\right)\right]^\beta + \int_0^t \left[\Phi\left(t, s, \zeta_\beta(s)\right)\right]^\beta ds, \right. \\ \left. \left[F\left(t, y_\beta^1(t_i)\right)\right]^\beta + \int_0^t \left[\Phi\left(t, s, x_\beta^1(t_i)\right)\right]^\beta ds\right) \le \bar{b}(m, \varepsilon, t, \eta_2),$$

where

$$\bar{b}(m, \varepsilon, t, \eta_2) = \lambda \frac{L}{m}(M_1 + M_2) + \left(\frac{L}{m}(M_1 + M_2) + \lambda\eta_2\right) \times \\ \times \int_0^t \mu(t, s) ds + \varepsilon (M_1 + M_2)\int_0^t \mu(t, s) ds .$$

Therefore,

$$\rho\left(\dot{\zeta}_{\beta}(t),\varepsilon\left[F\left(t,\zeta_{\beta}(t)\right)\right]^{\beta}+\varepsilon\int_{0}^{t}\left[\Phi\left(t,s,\zeta_{\beta}(s)\right)\right]^{\beta}ds\right)\le$$

$$\le\varepsilon\bar{b}\left(m,\varepsilon,t,\eta_{2}\right). \qquad (16)$$

From (16) it follows the existence of such a solution $x_{\beta}(t)$ of inclusion

$$\dot{x}_{\beta}\in\varepsilon\left[F\left(t,x_{\beta}\right)\right]^{\beta}+\varepsilon\int_{0}^{t}\left[\Phi\left(t,s,x_{\beta}(s)\right)\right]^{\beta}ds,\quad x_{\beta}(0)=x_{0},$$

that for all $t\in[0,L\varepsilon^{-1}]$

$$\left\|\zeta_{\beta}(t)-x_{\beta}(t)\right\|\le e^{(\lambda+M_{3})L}\bar{c}\left(m,\varepsilon,\eta_{2}\right),$$

where

$$\bar{c}\left(m,\varepsilon,\eta_{2}\right)=\left(M_{1}+M_{2}\right)\left(\frac{\lambda L^{2}}{m}+\varepsilon M_{4}L+\frac{L^{2}M_{3}}{m}\right)+$$

$$+L^{2}\eta_{2}\left(\lambda+M_{3}\right).$$

Than we obtain

$$\left\|y_{\beta}(t)-x_{\beta}(t)\right\|\le$$

$$\le4\left(M_{1}+M_{2}\right)\frac{L}{m}+L\eta_{2}+\bar{a}_{2}\left(m,\varepsilon\right)\left(e^{L(\lambda+\upsilon_{1})}-1\right)+$$

$$+e^{(\lambda+M_{3})L}\bar{c}\left(m,\varepsilon,\eta_{2}\right).$$

Also $\lim\limits_{\substack{\varepsilon\to0\\ m\to\infty}}\bar{a}_{2}\left(m,\varepsilon\right)=0$, and $\lim\limits_{\substack{\varepsilon\to0\\ m\to\infty}}\bar{c}\left(m,\varepsilon,\eta_{2}\right)=0$, then there exist m, ε, and η_2 such that $\left\|y_{\beta}(t)-x_{\beta}(t)\right\|<\eta$, for all $t\in[0,L\varepsilon^{-1}]$. Hence $\left[Y(t)\right]^{\beta}\subset\left[X(t)\right]^{\beta}+S_{\eta}(0)$, for all $t\in[0,L\varepsilon^{-1}]$. As $\beta\in[0,1]$ is arbitrary, then $Y(t)\subset X(t)+S_{\eta}(\hat{\theta})$, for all $t\in[0,L\varepsilon^{-1}]$. This concludes the proof.

5. Conclusions

In this article we prove the substantiation of the method of full averaging for the fuzzy integrodifferential inclusions with small parameter. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging. In a case if limits (5) and (6) do not exist it is possible to receive only result similar[46].

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