(1,2) - domination in Middle and Central Graph of $K_{1,n}$, $C_n$ and $P_n$

N. Murugesan, Deepa. S. Nair*

Post Graduate and Research Department of Mathematics Government Arts College, Coimbatore-18, India

Abstract In this paper we discuss domination and (1,2) - domination in middle graph and central graph of $K_{1,n}$, $C_n$ and $P_n$.

Keywords Dominating set, Domination Number, (1,2) - dominating Set, (1,2) - domination Number, Middle Graph, Central Graph

1. Introduction

Domination in graphs has become an important area of research in graph theory, as evidenced by the many results contained in the two books by Haynes,Hedetniemi and Slater(1998)[6]. Vernold Vivin J(2010) have studied the harmonious coloring of line graph, middle graph, central graphs of certain special graphs[12]. Venketakrishnan and Swaminathan (2010)[15] have studied the colorclass domination number of middle graph and central graphs of $K_{1,n}$, $C_n$ and $P_n$. In this paper we discuss (1,2)-domination in the middle and central graphs of $K_{1,n}$, $C_n$ and $P_n$.

By a graph $G=(V,E)$ we mean a finite undirected graph without loops or multiple edges. A subset $D$ of $V$ is a dominating set of $G$ if every vertex of $V-D$ is adjacent to a vertex of $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.

A (1,2) - dominating set in a graph $G$ is a set $S$ having the property that for every vertex $v$ in $V-S$ there is at least one vertex in $S$ at distance 1 from $v$ and a second vertex in $S$ at distance at most 2 from $v$. The order of the smallest (1,2) - dominating set of $G$ is called the (1,2) - domination number of $G$ denoted by $\gamma_{(1,2)}$.

For a given graph $G=(V,E)$ of order $n$, the central graph $C(G)$ is obtained, by subdividing each edge in $E$ exactly once and joining all the nonadjacent vertices of $G$. The central graph $C(G)$ of a graph $G$ is an example of a split graph, where a split graph is a graph whose vertex set $V$ can be partitioned into two sets, $V_1$ and $V_2$, where every pair of vertices in $V_1$ are adjacent, and no two vertices in $V_2$ are adjacent.

The middle graph $M(G)$ of a graph $G$, is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if they are either adjacent edges of $G$ or one is a vertex and the other is an edge incident with it. That is, two vertices $x$ and $y$ in the vertex set of $M(G)$ are adjacent if and only if $x,y$ are in $E(G)$ and $x,y$ are adjacent in $G$ or $x$ is in $V(G)$ and $y$ is in $E(G)$, and $x$ is incident to $y$ in $G$. The related ideas regarding these graphs can be seen in[3,12,13,14].

2. (1,2) - domination in Middle Graphs of $K_{1,n}$, $C_n$ and $P_n$.

Theorem 2.1

For any star graph $K_{1,n}$, $\gamma(M(K_{1,n}))=n$.

Proof: Let $V(K_{1,n})=\{v,v_1,v_2,\ldots,v_n\}$ and $E(G)=\{e_1,e_2,\ldots,e_n\}$ .By the definition of middle graph, we have $V[M(K_{1,n})]=\{v_i\cup\{e_i/1 \leq i \leq n\}\cup\{v_i/1 \leq i \leq n\}$ in which the vertices $e_1,e_2,\ldots,e_n,v$ induces a clique of order $n+1$. Hence $\gamma[M(K_{1,n})]\geq n$. But $\{v_1,v_2,\ldots,v_n\}$ is an independent set and each $e_i$ is adjacent to $v_i$. 

* Corresponding author: deepamtcr@gmail.com (Deepa. S. Nair)
Therefore \( \gamma[M(K_{1,n})] \leq n \)

Hence, \( \gamma[M(K_{1,n})] = n \).

**Theorem 2.2**

For any star graph \( K_{1,n} \), \((1,2)\)-domination number \( \gamma_{(1,2)}[M(K_{1,n})] \) = \( n \)

**Proof:**

Let \( V(K_{1,n}) = \{v_i, v_1, v_2, \ldots, v_n\} \) and \( E(G) = \{e_1, e_2, \ldots, e_n\} \). By the definition of middle graph, we have \( V[M(K_{1,n})] = \{v_i \cup \{e_i/1 \leq i \leq n\} \cup \{v_i/1 \leq i \leq n\} \}

in which the vertices \( e_1, e_2, \ldots, e_n,v \) induces a clique of order \( n+1 \). In \( M(K_{1,n}) \) the vertex \( v \) is adjacent to \( \{e_i/1 \leq i \leq n\\} \) and \( \{v_1, v_2, \ldots, v_n\} \) is an independent set and each \( e_i \) is adjacent to \( v_i \). So \( \{e_i/1 \leq i \leq n\} \) will form a \((1,2)\)-dominating set. That is, \( \gamma_{(1,2)}[M(K_{1,n})] = n \)

Since \( \{e_1, e_2, \ldots, e_n, v\} \) induces a clique, \( \gamma_{(1,2)}[M(K_{1,n})] \leq n \)

Therefore \( \gamma_{(1,2)}[M(K_{1,n})] = n \)

**Theorem 2.3**

For any cycle \( C_n \), \( \gamma[M(C_n)] = \left[ \frac{n}{3} \right] + 1 \)

**Proof:**

Let \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \) and \( E(C_n) = \{e_1, e_2, \ldots, e_n\} \)

where \( e_i = v_{i-1} v_i \) \( (1 \leq i \leq n-1) \), \( e_n = v_n v_1 \). By the definition of middle graph, \( M(C_n) \) has the vertex set \( V(C_n) \cup E(C_n) \) in which each \( e_i \) is adjacent with \( e_{i+1} (i = 1, 2, \ldots, n-1) \) and \( e_n \) is adjacent with \( v_1 \). In \( M(C_n) \), \( v_1, e_1, e_2, \ldots, e_{n-1}, v_1 \) induces a cycle of length \( 2n \).

But we know that for \( n \geq 3 \), \( \gamma(C_n) = \left[ \frac{n}{3} \right] \) (Theorem 2.1[5]).

Thus it is clear that \( \gamma[M(C_n)] = \left[ \frac{n}{3} \right] + 1 \).

**Theorem 2.4**

For any cycle \( C_n \), \( \gamma_{(1,2)}[M(C_n)] = \left[ \frac{n}{3} \right] + 1 \).

**Proof:**

Let \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \) and \( E(C_n) = \{e_1, e_2, \ldots, e_n\} \)

where \( e_i = v_i v_{i+1} \) \( (1 \leq i \leq n-1) \), \( e_n = v_n v_1 \). By the definition of middle graph, \( M(C_n) \) has the vertex set \( V(C_n) \cup E(C_n) \) in which each \( e_i \) is adjacent with \( e_{i+1} (i = 1, 2, \ldots, n-1) \) and \( e_n \) is adjacent with \( v_1 \). In \( M(C_n) \), \( v_1, e_1, e_2, \ldots, e_{n-1}, v_1 \) induces a cycle of length \( 2n \).

We have by theorem 3.1 in[10], for any cycle \( C_n \),

\( \gamma_{(1,2)}[M(C_n)] = \left[ \frac{n+2}{3} \right] \). Hence \( \gamma_{(1,2)}[M(C_n)] = \left[ \frac{n}{3} \right] + 1 \).
For any path \( P_n \), \( \gamma(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil \).

Proof:

Let \( P_n : v_1, v_2, v_3, \ldots, v_{n+1} \) be a path of length \( n \) and let \( v_{i+1} = e_i \). By the definition of middle graph, \( M(P_n) \) has the vertex set \( V(P_n) \cup E(P_n) = \{v_i : 1 \leq i \leq n+1\} \cup \{e_i : 1 \leq i \leq n\} \) in which each \( v_i \) is adjacent to \( e_i \) and \( e_i \) is adjacent to \( v_{i+1} \). Also \( e_i \) is adjacent to \( e_{i+1} \).

Case 1: \( |V(P_n)| = 2k \)

Then, the vertices \( v_1, e_1, v_2, e_2, \ldots, e_{2k-1}, v_k \) induces a path of length \( 4k \).

Case 2: \( |V(P_n)| = 2k + 1 \)

Then \( |V(M(P_n))| = 4k + 1 \). The vertices \( v_1, e_1, v_2, e_2, \ldots, e_{2k-1}, v_{2k+1} \) induces a path of length \( 4k \).

In both the cases, \( M(P_n) \) is a path of length \( 4k \). That is, \( n = 4k \).

But we have for \( n \geq 3 \), \( \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil \) (Theorem 2.1 [5]). Therefore, \( \gamma(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil \).

Theorem 2.6

For any path \( P_n \), \( \gamma_{(1,2)}(M(P_n)) = \left\lceil \frac{2n}{3} \right\rceil \).

Proof:

Let \( P_n : v_1, v_2, v_3, \ldots, v_{n+1} \) be a path of length \( n \) and let \( v_{i+1} = e_i \). By the definition of middle graph \( M(P_n) \) has the vertex set \( V(P_n) \cup E(P_n) = \{v_i : 1 \leq i \leq n+1\} \cup \{e_i : 1 \leq i \leq n\} \) in which each \( v_i \) is adjacent to \( e_i \) and \( e_i \) is adjacent to \( v_{i+1} \). Also \( e_i \) is adjacent to \( e_{i+1} \).

Case 1: \( |V(P_n)| = 2k \)

Then \( |V(M(P_n))| = 4k - 1 \). The vertices \( v_1, e_1, v_2, e_2, \ldots, e_{2k-1}, v_k \) induces a path of length \( 4k \).

Case 2: \( |V(P_n)| = 2k + 1 \)

Then \( |V(M(P_n))| = 4k + 1 \). The vertices \( v_1, e_1, v_2, e_2, \ldots, e_{2k-1}, v_{2k+1} \) induces a path of length \( 4k \).

In both the cases, \( M(P_n) \) is a path of length \( 4k \). That is, \( n = 4k \).

Then by theorem 2.1 in[10] we have \( \gamma_{(1,2)}(M(P_n)) = \left\lceil \frac{2n}{3} \right\rceil \).

3. \((1,2)-domination in Central Graph of \( K_{1,n}, C_n \) and \( P_n \)).

3.1. \((1,2)-domination in \( C(K_{1,n}) \))

Theorem 3.1

For any star graph \( K_{1,n} \), \( \gamma(C(K_{1,n})) = 2 \)

Proof:

Let \( V(K_{1,n}) = \{v_1, v_2, \ldots, v_n\} \) where \( \deg v = n \). By the definition of central graph of \( K_{1,n} \) we denote the vertices of subdivision by \( u_1, u_2, \ldots, u_n \). That is, \( v_1 \) is subdivided by \( u_i \) where \( 1 \leq i \leq n \). Let \( e_i = vu_i \) and \( e_i = vu_i \). Therefore \( V(C(K_{1,n})) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v\} \). By the definition of central graph the subgraph induced by the vertex set \( \{v_1, v_2, \ldots, v_n\} \) is \( K_n \) and let \( e_{ij} \) be the edge of \( C(K_{1,n}) \), connecting the vertex \( v_i \) and \( v_j \) where \( i < j \).

Therefore \( E(C(K_{1,n})) = \{e_{ij} : 1 \leq i \leq n\} \cup \{e_{ij} : 1 \leq i \leq n\} \cup \{e_{ij} \} \) \( \cup \{e_{ij} : 1 \leq i \leq n - 1, i + 1 \leq j \leq n\} \).

Since in the central graph of a star, the central vertex \( v \) together with any one of \( v_i \)'s form a dominating set. Therefore we have \( \gamma(C(K_{1,n})) = 2 \).

Central Graph of Star Graph \( K_{1,n} \)

Theorem 3.2

For any star graph \( K_{1,n} \), \( \gamma_{(1,2)}(C(K_{1,n})) = 2 \)

Proof:

Let \( V(K_{1,n}) = \{v_1, v_2, \ldots, v_n\} \) where \( \deg v = n \). By the definition of central graph of \( K_{1,n} \) we denote the
vertices of subdivision by $u_1, u_2, \ldots, u_n$. That is, $v_i$ is subdivided by $u_i (1 \leq i \leq n)$. Let $e_i = v_i u_i$ and $e'_i = v_i u_i (1 \leq i \leq n)$. Therefore

$$V(C(K_{1,n})) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v\}.$$  

By the definition of central graph the subgraph induced by the vertex set $\{v_1, v_2, \ldots, v_n\}$ is $K_n$ and let $e_{ij}$ be the edge of $C(K_{1,n})$, connecting the vertex $v_i$ and $v_j (i < j)$. Then

$$E(C(K_{1,n})) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{e_g : 1 \leq i \leq n - 1, i + 1 \leq j \leq n\}.$$  

Since in the central graph of a star, the central vertex of the star is adjacent to every vertex in the central graph the vertex $v$ together with any one of the vertices $u_i (1 \leq i \leq n)$ will form a $(1,2)$-dominating set. Hence $\gamma_{(1,2)}(C(K_{1,n})) = 2$.

### 3.2. $(1,2)$-domination in $C(C_d)$: Consider the Following Examples

- $\{v_1, u_2\}$ is a dominating set and also $(1,2)$-dominating set.  
  $\gamma(C(C_3)) = 2 = \gamma_{(1,2)}(C(C_3))$

- $\{v_1, u_2, u_3\}$ is a $(1,2)$-dominating set.  
  $\gamma(C(C_4)) = 3 = \gamma_{(1,2)}(C(C_4))$

- $\{v_1, u_2, u_3, u_4\}$ is a $(1,2)$-dominating set.  
  $\gamma(C(C_5)) = 4 = \gamma_{(1,2)}(C(C_5))$
\{v_1, u_2, u_3, u_4, u_5\} is a \((1,2)\)-dominating set.
\[\gamma(C(C_6)) = 5 = \gamma_{(1,2)}(C(C_6))\]

**Theorem 3.3**

For any cycle \(C_n\), \(\gamma(C(C_n)) = n - 1\)

**Proof:**

Let \(C_n\) be any cycle of length \(n\) and let
\[V(C_n) = \{v_1, v_2, \ldots, v_n\}\]
and
\[E(C_n) = \{e_1, e_2, \ldots, e_n\}\].

By the definition of central graph \(C(C_n)\) has the vertex set \(V(C_n) \cup \{u_i : 1 \leq i \leq n\}\) where \(u_i\) is a vertex of subdivision of the edge \(v_iv_{i+1}\) \((1 \leq i \leq n-1)\) and \(u_n\) is a vertex of subdivision of the edge \(v_nv_1\). In \(C(C_n)\) we can note that the vertex \(v_i\) is adjacent with all vertices except the vertices \(v_{i+1}\) and \(v_{i-1}\) for \(1 \leq i \leq n-1\). \(v_1\) is adjacent with all vertices except \(v_{n-1}\) and \(v_1\). The total number of edges incident with \(v_i\) is \((n-1)\) for every \(i = 1, 2, \ldots, n\) and \(\{u_i, 1 \leq i \leq n\}\) is an independent set. So \(\{v_i\} \cup \{u_i / 2 \leq i \leq n - 1\}\)

will be a dominating set. So the dominating set will consist of \((n-1)\) vertices.

Hence \(\gamma(C(C_n)) = n - 1\)

**Theorem 3.4**

For any cycle \(C_n\), \(\gamma_{(1,2)}(C(C_n)) = n - 1\)

**Proof:**

Let \(C_n\) be any cycle of length \(n\) and let
\[V(C_n) = \{v_1, v_2, \ldots, v_n\}\]
and
\[E(C_n) = \{e_1, e_2, \ldots, e_n\}\].

By the definition of central graph \(C(C_n)\) has the vertex set \(V(C_n) \cup \{u_i : 1 \leq i \leq n\}\) where \(u_i\) is a vertex of subdivision of the edge \(v_iv_{i+1}\) \((1 \leq i \leq n-1)\) and \(u_n\) is a vertex of subdivision of the edge \(v_nv_1\). In \(C(C_n)\) we can note that the vertex \(v_i\) is adjacent with all vertices except the vertices \(v_{i+1}\) and \(v_{i-1}\) for \(1 \leq i \leq n-1\). \(v_1\) is adjacent with all vertices except \(v_{n-1}\) and \(v_1\). The total number of edges incident with \(v_i\) is \((n-1)\) for every \(i = 1, 2, \ldots, n\) and \(\{u_i, 1 \leq i \leq n\}\) is an independent set. So \(\{v_i\} \cup \{u_i / 2 \leq i \leq n - 1\}\)

will be a dominating set. But every minimum cardinality dominating set is also a \((1,2)\)-dominating set in \(C(C_n)\)

Hence \(\gamma_{(1,2)}(C(C_n)) = n - 1\)
3.3. (1,2)-domination in $C(P_n)$

$v_{1, u_1}$ is a dominating set and also (1,2)-dominating set.

$$\gamma(C(P_3)) = 2 = \gamma_{(1,2)}(C(P_3)).$$

$P_4$:

$\{v_1, u_2, u_3\}$ is a (1,2)-dominating set.

$$\gamma(C(P_4)) = 3 = \gamma_{(1,2)}(C(P_4)).$$

$P_5$:

$\{v_1, u_2, u_3, u_4\}$ is a (1,2)-dominating set.

$$\gamma(C(P_5)) = 4 = \gamma_{(1,2)}(C(P_5)).$$

**Theorem 3.5**

For any path $P_n$ $\gamma(C(P_n)) = n-1$

**Proof.**

Let $P_n$ be any path of length $n-1$ with vertices $v_1, v_2, \ldots, v_n$. On the process of centralisation of $P_n$, let $u_i$ be the vertex of subdivision of the edges $v_i v_{i-1}$ ($1 \leq i \leq n$).
Also let \( v_i u_i = e_i \) and \( u_i v_{i+1} = e'_i \) (\( 1 \leq i \leq n-1 \)).

By the definition of central graph the non-adjacent vertices \( v_i \) and \( v_j \) of \( P_n \) are adjacent in \( C(P_n) \) by the edge \( e_{ij} \). Therefore \( V(C(P_n)) = \{v_i / 1 \leq i \leq n\} \cup \{u_i / 1 \leq i \leq n-1\} \) and \( E(C(P_n)) = \{e_i : 1 \leq i \leq n-1\} \cup \{e'_i : 1 \leq i \leq n-2, i+2 \leq j \leq n\} \). In \( C(P_n) \) we can see that the vertex \( v_1 \) is adjacent with all vertices except the vertices \( v_{1+j} \) and \( v_{i+1} \) for \( 1 \leq i \leq n-1 \). \( v_1 \) is adjacent with all vertices except \( v_2 \). \( \{u_i : 1 \leq i \leq n\} \) is an independent set. So \( \{v_1\} \cup \{u_i / 2 \leq i \leq n-1\} \). So \( \{v_i\} \cup \{u_i / 2 \leq i \leq n-1\} \) will be a dominating set.

\[ \gamma(C(P_n)) = n-1. \]

**Theorem 3.6**

For any path \( P_n, \gamma((1,2) C(P_n)) = n-1. \)

**Proof:**

Let \( P_n \) be any path of length \( n-1 \) with vertices \( v_1, v_2, \ldots, v_n \). On the process of centralisation of \( P_n \), let \( u_i \) be the vertex of subdivision of the edges \( v_i v_{i-1} \) (\( 1 \leq i \leq n \)).

Also let \( v_i u_i = e_i \) and \( u_i v_{i+1} = e'_i \) (\( 1 \leq i \leq n-1 \)).

By the definition of central graph the non-adjacent vertices \( v_i \) and \( v_j \) of \( P_n \) are adjacent in \( C(P_n) \) by the edge \( e_{ij} \). Therefore \( V(C(P_n)) = \{v_i / 1 \leq i \leq n\} \cup \{u_i / 1 \leq i \leq n-1\} \) and \( E(C(P_n)) = \{e_i : 1 \leq i \leq n-1\} \cup \{e'_i : 1 \leq i \leq n-2, i+2 \leq j \leq n\} \). In \( C(P_n) \) we can see that the vertex \( v_1 \) is adjacent with all vertices except the vertices \( v_{1+j} \) and \( v_{i+1} \) for \( 1 \leq i \leq n-1 \). \( v_1 \) is adjacent with all vertices except \( v_2 \). \( \{u_i : 1 \leq i \leq n\} \) is an independent set. So \( \{v_1\} \cup \{u_i / 2 \leq i \leq n-1\} \). So \( \{v_i\} \cup \{u_i / 2 \leq i \leq n-1\} \) will be a dominating set.

\[ \gamma(C(P_n)) = n-1. \]

But every minimum cardinality dominating set of the central graph of a path is also a (1,2)-dominating set. Hence \( \gamma((1,2) C(P_n)) = n-1. \)

**4. Relation between Domination Number and (1,2)-domination Number in Middle and Central Graph of Stars, Cycles and Paths**

**Theorem 4.1**

In the middle graph of a star, \( [M(K_{1,n})] \), the domination number equals the (1,2)-domination number.

**Proof:**

This result is obvious from theorem 2.1 and 2.2.

**Theorem 4.2**

In the middle graph of cycles, \( [M(C_n)] \), the domination number equals the (1,2)-domination number.

**Proof:**

This result is obtained by theorem 2.3 and theorem 2.4.

**Theorem 4.3**

In the middle graph of paths, \( [M(P_n)] \), the domination number is less than or equal to the (1,2)-domination number.

**Proof:**

This result is obtained by theorem 2.5 and theorem 2.6.

**Theorem 4.4**

In the central graph of any star, \( C(K_{1,n}) \), the domination number equals the (1,2)-domination number.

**Proof:**

This is clear from theorem 3.1 and 3.2.

**Theorem 4.5**

In the central graph of cycles, \( C(C_n) \), the domination number equals the (1,2)-domination number.

**Proof:**
This result is due to the theorem 3.3 and 3.4.

**Theorem 4.6**

In the central graph of paths, $C(P_n)$, the domination number equals the $(1,2)$-domination number.

**Proof:**

This result is obtained from theorem 3.5 and 3.6.

**5. Conclusions**

In this paper we have extended $(1,2)$-domination to the middle graph and central graph of stars, cycles, and paths and discussed both domination and $(1,2)$-domination number of these graphs. In all cases it is important to see that the domination number is less than or equal to the $(1,2)$-domination which coincides the result established in [8].

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