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Approximate Solution to Burgers Equation Using Reconstruction of Variational Iteration Method

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Abstract In this Letter, we present the reconstruction of variational iteration method (RVIM) for obtaining the analytical solution of the one-dimensional nonlinear Burgers' equation. The initial approximation can be freely chosen with possible unknown constants which can be determined by imposing the boundary and initial conditions. Convergence of the solution and effects for the method is discussed. Burgers equation is used for describing wave processes in acoustics and hydrodynamics. It illustrates the validity and the great potential of the reconstruction of variational iteration method in solving nonlinear partial differential equations. The obtained results reveal that the technique introduced here is very effective and convenient for solving nonlinear partial differential equations and nonlinear ordinary differential equations that we are found to be in good agreement with the exact solutions.

Keywords Reconstruction of Variational Iteration Method(RVIM), Burgers Equation, Approximate Solution

1. Introduction

Most phenomena in real world are described through nonlinear equations and these types of equations have attracted lots of attention among scientists. Large classes of nonlinear equations do not have a precise analytic solution, so numerical methods have largely been used to handle these equations. There are also some analytic techniques for nonlinear equations. Some of the classic analytic methods are the Lyapunov's artificial small parameter method, perturbation techniques and d-expansion method. In the last two decades, some new analytic methods have been proposed to handle functional equations; among them are Adomian decomposition method (ADM)[1,2], homotopy analysis method (HAM)[3-8], variation iteration method (VIM)[9-12] and homotopy perturbation method (HPM)[13-28]. Burgers' model of turbulence is a very important fluid dynamic model and the study of this model and the theory of shock waves have been considered by many authors both for conceptual understanding of a class of physical flows and for testing various numerical methods. It is still a hot spot to seek new methods to obtain new exact or approximate solutions[29–43]. The Burgers' equation is used as a model in fields as wide as Acoustics, continues stochastic processes, dispersive water waves, gas dynamics, heat conduction, longitudinal elastic waves in an isotropic solid, number theory, shock waves, turbulence. For that different methods have been put forward to seek various exact solutions of multifarious physical models described by nonlinear PDEs. The well-known model was first introduced by Bateman[6] who found its steady solutions, descriptive of certain viscous flows. It was later proposed by Burgers[7] as one of a class of equations describing mathematical models of turbulence. In the context of gas dynamics, it was discussed by Hopf[8] and Cole[9]. They also illustrated independently that the Burgers' equation can be solved exactly for an arbitrary initial condition. Benton and Platzman[10] have surveyed the analytical solutions of the one-dimensional Burgers' equation. It can be considered as a simplified form of the Navier–Stokes equation[11] due to the form of nonlinear convection term and the occurrence of the viscosity term. Numerical algorithms for the solution of Burgers' equation have been propsosed by many authors. In this Letter, we investigate the solving the Burgers' equation. We also give a framework for theoretical analysis of the RVIM. This scheme will be illustrated by studying the Burgers' equation to compute explicit and numerical solutions.

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2. Description of the Method

In the following section,an alternative method for finding the optimal value of the Lagrange multiplier by the use of the Laplace transform[15] will be investigated a large of problems in science and engineering involve the solution of partial differential equations. Suppose x, t are two independent variables; consider t as the principal variable and x as the secondary variable. If $u(x,t)$ is a function of two variables x and t, when the Laplace transform is applied with t as a variable, definition of Laplace transform is

$$\mathbb{L}[u(x,t);s] = \int_0^{\infty} e^{-st}\, u(x,t)dt \quad (1)$$

We have some preliminary notations as

$$\mathbb{L}\left[\frac{\partial u}{\partial t};s\right] = \int_0^{\infty} e^{-st}\frac{\partial u}{\partial t}\,dt = sU(x,s) - u(x,0) \quad (2)$$

$$\mathbb{L}\left[\frac{\partial^2 u}{\partial t^2};s\right] = s^2U(x,s) - su(x,0) - u_t(x,0) \quad (3)$$

Where

$$U(x,s) = \mathbb{L}[u(x,t);s] \quad (4)$$

We often come across functions which are not the transform of some known function, but then, they can possibly be as a product of two functions, each of which is the transform of a known function. Thus we may be able to write the given function as $U(x,s), V(x,s)$ where $U(s)$ and $V(s)$ are known to the transform of the functions $u(x,t), v(x,t)$ respectively. The convolution of $u(x,t)$ and $v(x,t)$ is written $u(x,t) * v(x,t)$. It is defined as the integral of the product of the two functions after one is reversed and shifted.

Convolution Theorem: if $U(x,s), V(x,s)$ are the Laplace transform of $u(x,t), v(x,t)$, when the Laplace transform is applied to t as a variable, respectively; then $U(x,s).V(x,s)$ is the Laplace Transform of $\int_0^t u(x,t-\mathcal{E})v(x,\mathcal{E})d\mathcal{E}$

$$\mathbb{L}^{-1}[U(x,s).V(x,s)] = \int_0^t u(x,t-\mathcal{E})v(x,\mathcal{E})d\mathcal{E} \quad (5)$$

To facilitate our discussion of Reconstruction of Variational Iteration Method, introducing the new linear or nonlinear function $h(u(t,x)) = f(t,x) - N(u(t,x))$ and considering the new equation, rewrite $h(u(t,x)) = f(t,x) - N(u(t,x))$ as

$$L(u(t,x)) = h(t,x,u)) \quad (6)$$

Now, for implementation the correctional function of VIM based on new idea of Laplace transform, applying Laplace Transform to both sides of the above equation so that we introduce artificial initial conditions to zero for main problem, then left hand side of equation after transformation is featured as

$$\mathbb{L}[L\{u(x,t)\}] = U(x,s)P(s) \quad (7)$$

Where $P(s)$ is polynomial with the degree of the highest order derivative of the selected linear operator.

$$\mathbb{L}[L\{u(x,t)\}] = U(x,s)P(s) = \mathbb{L}[h\{(x,t,u)\}] \quad (8)$$

Then

$$U(x,s) = \frac{\mathbb{L}[h\{(x,t,u)\}]}{P(s)} \quad (9)$$

Suppose that $D(s) = \frac{1}{P(s)}$, and $\mathbb{L}[h\{(x,t,u)\}] = H(x,s)$. Therefore using the convolution theorem we have

$$U(x,s) = D(s).H(x,s) = \mathbb{L}\{(d(t) * h(x,t,u))\} \quad (10)$$

Taking the inverse Laplace transform on both side of Eq.

$$u(x,t) = \int_0^t d(t-\mathcal{E})h(x,\mathcal{E},u)d\mathcal{E} \quad (11)$$

Thus the following reconstructed method of variational iteration formula can be obtained

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t d(t-\mathcal{E})h(x,\mathcal{E},u_n)d\mathcal{E} \quad (12)$$

And $u_0(x,t)$ is initial solution with or without unknown parameters. In absence of unknown parameters, $u_0(x,t)$ should satisfy initial/ boundary conditions.

3. Application of the Proposed Method

The mentioned method (RVIM) is able to solve a wide range of linear and nonlinear equations.

In this paper to illustrate basic concepts of Reconstructed of Variational Iteration Method[15] as it seen in following we concentrated on solution of burgers equation that placed in linear equations classes.

$$u_t + uu_x = u_{xx} \quad (13)$$

Subject to the initial conditions:

$$u(x,\emptyset) = \frac{1}{2} - \frac{1}{2}tanh(\frac{x}{4}) \quad (14)$$

The exact solution for this problem is

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh(\frac{1}{4}(x - \frac{1}{2}t)) \quad (15)$$

At first rewrite eq.(13) based on selective linear operator as

$$\mathbb{L}\{u(x,t)\} = u_t = \overbrace{u_{xx} - uu_x}^{h(x,t,u)} \quad (16)$$

Now Laplace transform is implemented with respect to independent variable t on both sides of eq.(16) by using the new artificial initial condition (which all of them are zero) and we have

$$s \cup (x,t) = \mathbb{L}\{h(x,t,u)\} \quad (17)$$

$$\cup (x,t) = \frac{\mathbb{L}\{h(x,t,u)\}}{s} \quad (18)$$

And whereas Laplace inverse transform of $1/s^2$ is as follows

$$\mathbb{L}^{-1}\left[\frac{1}{s}\right] = 1 \quad (19)$$

Therefore by using the Laplace inverse transform and convolution theorem is conclude that

$$u(x,t) = \int_0^t h(x,\varepsilon,u)\,d\varepsilon \quad (20)$$

Hence, we arrive the following iterative formula for the approximate solution of subject to the initial condition (14).

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t u_{xx}(x,\varepsilon) - u(x,\varepsilon) * uxx,\varepsilon d\varepsilon \quad (21)$$

Now it is assumed that an initial approximation has the form

$$u(x,\emptyset) = \frac{1}{2} - \frac{1}{2}\tanh(\frac{x}{4}) \quad (22)$$

Therefore, the following first-order approximate solution is derived:

$$u_1(x,t) = \frac{1}{2} - \frac{1}{2}\tanh(\frac{x}{4}) + \frac{1}{4}\tanh\left(\frac{x}{4}\right)\left((\frac{1}{4} - \frac{1}{4}\tanh(\frac{x}{4})^2)t - (\frac{1}{2} - \frac{1}{2}\tanh(\frac{x}{4})\right)(-\frac{1}{8} + \frac{1}{8}\tanh\left(\frac{x}{4}\right)^2)t \quad (23)$$

Table 1. comparison between errors of u_1 and u_3

X	t=0.2		t=0.6		t=0.9		t=1.2		t=1.5	
	e_{u_1}	e_{u_3}	e_{u_1}	e_{u_3}	e_{u_1}	e_{u_3}	e_{u_1}	e_{u_3}	e_{u_1}	e_{u_3}
0.2	2.66E-05	1.70E-07	1.36E-04	1.35E-05	1.48E-04	6.75E-05	6.77E-07	2.11E-04	-3.81E-04	5.09E-04
0.7	1.17E-04	1.65E-07	9.08E-04	1.30E-05	1.80E-03	6.51E-05	2.81E-03	2.02E-04	3.81E-03	4.88E-04
1	1.79E-04	1.42E-07	1.43E-03	1.11E-05	2.92E-03	5.55E-05	4.71E-03	1.72E-04	6.66E-03	4.14E-04
1.5	2.89E-04	7.12E-08	2.36E-03	5.56E-06	4.94E-03	2.75E-05	8.14E-03	8.51E-05	1.17E-02	2.04E-04
1.7	3.35E-04	3.46E-08	2.75E-03	2.67E-06	5.77E-03	1.31E-05	9.56E-03	4.07E-05	1.39E-02	9.77E-05
2	4.04E-04	-2.64E-08	3.33E-03	-2.01E-06	7.03E-03	-9.93E-06	1.17E-02	-3.04E-05	1.71E-02	-7.19E-05
2.4	4.95E-04	-1.06E-07	4.11E-03	-8.28E-06	8.70E-03	-4.06E-05	1.45E-02	-1.24E-04	2.13E-02	-2.94E-04
2.7	5.61E-04	-1.63E-07	4.67E-03	-1.25E-05	9.91E-03	-6.12E-05	1.66E-02	-1.87E-04	2.44E-02	-4.42E-04
3	6.24E-04	-2.08E-07	5.21E-03	-1.60E-05	1.10E-02	-7.82E-05	1.86E-02	-2.38E-04	2.74E-02	-5.62E-04

Table 2. comparison between errors of u_3 and u_5

X	t=0.2		t=0.6		t=0.9		t=1.2		t=1.5	
	e_{u_3}	e_{u_5}	e_{u_3}	e_{u_5}	e_{u_3}	e_{u_5}	e_{u_3}	e_{u_5}	e_{u_3}	e_{u_5}
0.2	1.70E-07	2.66E-10	1.35E-05	1.61E-07	6.75E-05	9.91E-06	2.11E-04	9.91E-06	5.09E-04	3.71E-05
0.7	1.65E-07	4.69E-11	1.30E-05	3.25E-08	6.51E-05	1.24E-06	2.02E-04	1.24E-06	4.88E-04	3.34E-06
1	1.42E-07	-2.56E-10	1.11E-05	-8.40E-08	5.55E-05	-6.11E-06	1.72E-04	-6.11E-06	4.14E-04	-2.46E-05
1.5	7.12E-08	-6.02E-10	5.56E-06	-2.55E-07	2.75E-05	-1.63E-05	8.51E-05	-1.63E-05	2.04E-04	-6.22E-05
1.7	3.46E-08	-9.67E-10	2.67E-06	-2.93E-07	1.31E-05	-1.83E-05	4.07E-05	-1.83E-05	9.77E-05	-6.91E-05
2	-2.64E-08	-3.58E-10	-2.01E-06	-3.03E-07	-9.93E-06	-1.83E-05	-3.04E-05	-1.83E-05	-7.19E-05	-6.82E-05
2.4	-1.06E-07	-4.15E-10	-8.28E-06	-2.31E-07	-4.06E-05	-1.34E-05	-1.24E-04	-1.34E-05	-2.94E-04	-4.88E-05
2.7	-1.63E-07	0	-1.25E-05	-1.36E-07	-6.12E-05	-7.41E-06	-1.87E-04	-7.41E-06	-4.42E-04	-2.60E-05
3	-2.08E-07	0	-1.60E-05	-2.81E-08	-7.82E-05	-8.83E-07	-2.38E-04	-8.82E-07	-5.62E-04	-1.77E-06

In table 1, the amounts of error for u_1 and u_3 are expressed. The error of u has strong relevance to two independent parameters, x and t. x range is chosen between 0.2 to 3 and t is between 0.2 to 1.5, error depends on various amounts of x and t. as a mention the difference between errors of u_1 and u_3 is noticeable.

As it is shown in table 1. The amounts of errors for $u_3$and u_5 are illustrated in table2. like last state the amounts of errors depend on parameters x and t. to simulate more and show good enhancement in result, the ranges of x and t are chosen similar to the previous table (table 1.).as it is seen in comparison to table 1. the amounts of errors are showing a great reduction and this reduction is considerable between u_3 andu_5.u1 u3 u5 and the tables above are selected in a way to demonstrate the accuracy, efficiency and authenticity of the method for sequent stages.

4. Conclusions

In this Letter, reconstruction of variational iteration method has been successfully applied to find the solution of burgers equation. The results of the present method are in excellent agreement with some earlier works and the obtained solutions are shown in tables[1-**]. Reconstruction of Variational Iteration Method gives several successive approximations through using the iteration of the correction functional. In our work, we use the Maple Package for our calculations. Some of the advantages of RVIM are its fast rapidly convergent series and high accuracy.

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