Weak Insertion of a $\gamma$–Continuous Function

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Abstract  A sufficient condition in terms of lower cut sets are given for the weak insertion of a $\gamma$–continuous function between two comparable real-valued functions.

Keywords  Weak Insertion, Strong Binary Relation, Preopen Set, Semi-Open Set, $\gamma$–Open Set, Lower Cut Set

1. Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964[5]. A subset $A$ of a topological space $(X, \tau)$ is called preopen or locally dense or nearly open if $A \subseteq \text{Int(Cl}(A))$. A set $A$ is called preclosed if its complement is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subseteq A$. The term ,preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb[13], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael[5].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963[12]. A subset $A$ of a topological space $(X, \tau)$ is called semi-open[12] if $A \subseteq \text{Cl}(\text{Int}(A))$. A set $A$ is called semi-closed if its complement is semi-open or equivalently if $\text{Int}(\text{Cl}(A)) \subseteq A$. A property $P$ defined on a topological space $(X, \tau)$ is called $\gamma$–open if $A \cap S$ is semi-open whenever $S$ is semi-open. A set $A$ is called $\gamma$–closed if its complement is $\gamma$–open or equivalently if $A \cup S$ is preclosed, whenever $S$ is preclosed. The class $\gamma$–open sets is a topology on $X[1]$. A real-valued function $f$ defined on a topological space $X$ is called $\gamma$–continuous if the preimage of every open subset of $X$ belongs to $A$, where $A$ is a collection of subset of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $\gamma$–continuity. However, for unknown concepts the reader may refer to[6,7].

Hence, a real-valued function $f$ defined on a topological space $X$ is called $\gamma$–continuous[14] if the preimage of every open subset of $R$ belongs to $A$, where $A$ is a collection of subset of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $\gamma$–continuity. However, for unknown concepts the reader may refer to[6,7].

Before giving a sufficient condition for insertability of a $\gamma$–continuous function, the necessary definitions and terminology are stated.

Let $(X, \tau)$ be a topological space, the family of all $\gamma$–open, $\gamma$–closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\gamma\text{O}(X, \tau)$, $\gamma\text{C}(X, \tau)$, $\text{sO}(X, \tau)$, $\text{sC}(X, \tau)$, $\text{pO}(X, \tau)$ and $\text{pC}(X, \tau)$, respectively.

Definition 2.1. Let $A$ be a subset of a topological space $(X, \tau)$. Respectively, we define the $\gamma$–closure, $\gamma$–interior, $\text{s}$–closure, $\text{s}$–interior, $\text{p}$–closure and $\text{p}$–interior of a set $A$, denoted by $\gamma\text{Cl}(A)$, $\gamma\text{Int}(A)$, $\text{sCl}(A)$, $\text{sInt}(A)$, $\text{pCl}(A)$ and $\text{pInt}(A)$ as follows:

$$\gamma\text{Cl}(A) = \{ F : F \supseteq A, F \in \gamma\text{C}(X, \tau) \}.$$
\[
\gamma \text{Int}(A) = \bigcup \{ O : O \subseteq A, O \in \gamma \text{O}(X, t) \}, \\
s\text{Cl}(A) = \bigcap \{ F : F \supseteq A, F \in \text{sCl}(X, t) \}, \\
s\text{Int}(A) = \bigcup \{ O : O \subseteq A, O \in \text{sO}(X, t) \}, \\
p\text{Cl}(A) = \bigcap \{ F : F \supseteq A, F \in \text{pCl}(X, t) \} \\
\text{Cl}(A) = \bigcup \{ O : O \subseteq A, O \in \text{pCl}(X, t) \}.
\]

If \( \{ A_i : i \in I \} \) be a family of preopen (resp. semi-open) sets, since \( A_i \subseteq \text{Int}(\{ A_i : i \in I \}) \) (resp. \( A_i \subseteq \text{Cl}(\{ A_i : i \in I \}) \)), then \( \bigcup \{ A_i : i \in I \} \subseteq \text{Cl}(\{ A_i : i \in I \}) \) (resp. \( \bigcup \{ A_i : i \in I \} \subseteq \text{Cl}(\{ A_i : i \in I \}) \)), i.e., \( \bigcup \{ A_i : i \in I \} \) is a preopen (resp. semi-open) set. Therefore, both preopen and semi-open sets are preserved by arbitrary unions.

Hence, respectively, we have \( \gamma \text{Cl}(A), s\text{Cl}(A), p\text{Cl}(A) \) are \( \gamma \)-closed, semi-closed, and \( \gamma \)-open sets, \( s\text{Int}(A), p\text{Int}(A) \) are \( \gamma \)-open, semi-open, and \( \gamma \)-preopen sets, respectively.

The following first two definitions are modifications of conditions considered in [9, 10].

**Definition 2.2.** If \( \rho \) is a binary relation in a set \( S \) then \( \rho^- \) is defined as follows: \( x \rho^- y \) if and only if \( y \rho x \) implies \( x \rho y \) and \( u \rho x \) implies \( u \rho y \) for any \( u \in S \).

**Definition 2.3.** A binary relation \( \rho \) in the power set \( P(X) \) of a topological space \( X \) is called a strong binary relation in \( P(X) \) in case \( \rho \) satisfies each of the following conditions:

1) If \( A_i \rho B_j \) for any \( i \in \{ 1, ..., m \} \) and for any \( j \in \{ 1, ..., n \} \), then there exists a set \( C \in P(X) \) such that \( A_i \rho C \) and \( C \rho B_j \) for any \( i \in \{ 1, ..., m \} \) and any \( j \in \{ 1, ..., n \} \).

2) If \( A \subseteq B \subseteq A \), then \( \rho^- B \).

3) If \( A \rho B \), then \( \gamma \text{Cl}(A) \subseteq B \) and \( A \subseteq \gamma \text{Int}(B) \).

The concept of a lower indefinite cut set for a real number \( t \) is defined by Brooks [4] as follows:

**Definition 2.4.** If \( f \) is a real-valued function defined on a space \( X \) and if \( \{ x \in X : f(x) < t \} \subseteq \{ x \in X : f(x) > t \} \) for a real number \( t \), then \( A(f, t) \) is called a lower indefinite cut set in the domain of \( f \) at the level \( t \) of the function.

We now give the following main result:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions defined on a topological space \( X \) with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the power set of \( X \) and if there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f, t_1) \rho A(g, t_2) \), then there exists a \( \gamma \)-continuous function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

**Proof.** Let \( g \) and \( f \) be real-valued functions defined on \( X \) such that \( g \leq f \). By hypothesis there exists a strong binary relation \( \rho \) on the power set of \( X \) and there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f, t_1) \rho A(g, t_2) \).

Define functions \( F \) and \( G \) mapping the rational numbers \( \mathbb{Q} \) into the power set of \( X \) by \( F(t) = A(f, t) \) and \( G(t) = A(g, t) \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then \( F(t_1) \rho F(t_2) \), \( G(t_1) \rho G(t_2) \), and \( H(t_1) \rho H(t_2) \). By Lemmas 1 and 2 of [10] it follows that there exists a function \( H \) mapping \( \mathbb{Q} \) into the power set of \( X \) such that if \( t_1 \) and \( t_2 \) are any rational numbers with \( t_1 < t_2 \), then \( F(t_1) \rho H(t_2), H(t_1) \rho H(t_2) \) and \( H(t_1) \rho H(t_2) \).

For any \( x \in X \), let \( h(x) = \inf \{ t \in \mathbb{Q} : f(x) < t \} \).

We first verify that \( g \leq h \leq f \). If \( x \) is in \( H(t_1) \) then \( x \) is in \( G(k) \) for any \( k < t \); since \( x \) is in \( G(k) = A(f, k) \) implies that \( g(x) \leq k \), it follows that \( g(x) \leq \inf \{ t \in \mathbb{Q} : f(x) < t \} \). Hence \( g \leq h \).

Also, \( \rho \) for any rational numbers \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \), we have \( h(t_1, t_2) = \gamma \text{Int}(H(t_2)) \cap \gamma \text{Cl}(H(t_1)) \). Hence \( h(t_1, t_2) \) is a \( \gamma \)-open subset of \( X \), i.e., \( h \) is a \( \gamma \)-continuous function on \( X \).

The above proof used the technique of proof of Theorem 1 of [9].

**3. Applications**

The abbreviations \( pc \) and \( sc \) are used for precontinuous and semicontinuous, respectively.

Before stating the consequences of Theorem 2.1, we suppose that \( X \) is a topological space that \( \gamma \)-open sets are semi-open and preopen.

**Corollary 3.1.** If for each pair of disjoint preclosed (resp. semi-closed) sets \( F_1, F_2 \), there exist \( \gamma \)-open sets \( G_1 \) and \( G_2 \) such that \( F_1 \subseteq G_1, F_2 \subseteq G_2 \) and \( G_1 \cap G_2 = \emptyset \) then every precontinuous (resp. semi-continuous) function is \( \gamma \)-continuous.

**Proof.** First verify that \( X \) has the weak \( \gamma \)-insertion property for (pc, pc) (resp. (sc, sc)): Let \( g \) and \( f \) be real-valued functions defined on the \( X \), such that \( f \) and \( g \) are \( pc \) (resp. \( sc \)), and \( g \leq f \). If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( s\text{Cl}(A) \subseteq B \), then \( h(t_1, t_2) \) is a binary relation \( \rho \) on the power set of \( X \), such that \( F_1 \subseteq G_1, F_2 \subseteq G_2 \) and \( G_1 \cap G_2 = \emptyset \) then every precontinuous (resp. semi-continuous) function is \( \gamma \)-continuous.

**Proof.** Let \( g \) and \( f \) be real-valued functions functions defined on the \( X \), such that \( g \) is \( pc \) (resp. \( sc \)), and \( g \leq f \). If a binary relation \( \rho \) is defined by \( A \rho B \) in case \( s\text{Cl}(A) \subseteq B \), then \( h(t_{1 < t_2} \) is a lower \( \gamma \)-open subset of \( X \), i.e., \( h \) is a \( \gamma \)-continuous function on \( X \).

The above proof used the technique of proof of Theorem 1 of [9].
since \( \{ x \in X : f(x) \leq t_1 \} \) is a semi-closed (resp. preclosed) set and since \( \{ x \in X : g(x) < t_2 \} \) is a preopen (resp. semi-open) set, it follows that \( \text{sCl}(A(f, t_1)) \subseteq \text{plnt}(A(g, t_2)) \) (resp. \( \text{pCl}(A(f, t_1)) \subseteq \text{sInt}(A(g, t_2)) \)). Hence \( t_1 < t_2 \) implies that \( A(f, t_1) \approx A(g, t_2) \). The proof follows from Theorem 2.1.

Remark 3.1. See[1,2], for examples of topological spaces are said in corollaries 3.1 and 3.2.

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**REFERENCES**


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