

On a Problem with Conditions on All Boundary for a Pseudoparabolic Equation

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Abstract In the paper, the considered equation as a pseudoparabolic equation generalizes not only classic equations of mathematical physics (heat-conductivity equations, string vibration equation) and also many models differential equations (telegraph equation, Aller's equation, moisture transfer generalized equation, Manjeron equation, Boussinesq-Lyav equation and etc.). For this pseudoparabolic differential equation of sixth order with discontinuous coefficients we consider a problem with conditions on all the boundary. As a non-local problem this problem by its statement plays an intermediate role between the problems with boundary conditions of first kind (i.e. Dirichlet problems) and second kind (i.e. Neumann problems) and differs from them. Furthermore, it is easy to see that by its statement this non-local problem doesn't coincide with the third boundary value problems and very differs from these problems. It is grounded that the boundary condition on all the boundary in the classic and non-classic treatment are equivalent to each other, and such boundary conditions are demonstrated in geometric form. Even from geometric interpretation can see that the grounded non-classic treatment doesn't require any additional conditions of agreement type. Thus, namely in this paper, the non-classic problem with conditions on all the boundary is grounded for a pseudoparabolic equation of sixth order. For simplicity, this was demonstrated for one model case in one of S.L. Sobolev's isotropic space $W_p^{(3,3)}(G)$.

Keywords Non-local Boundary Problem, a Problem with Conditions on all the Boundary, Pseudoparabolic Equation, Discontinuous Coefficients Equations

1. Introduction

Urgency of investigations conducted in this field is explained by appearance of local and non-local problems for discontinuous coefficients equations connected with different applied problems. Such type problems arise for example, while studying the problems of moisture, transfer in soils, heat transfer in heterogeneous media, diffusion of thermal neutrons in inhibitors, simulation of different biological processes, phenomena and etc[1-7].

Pseudoparabolic equations are attracted for sufficiently adequate description of a great deal of real processes occurring in the nature, engineering and etc. In particular, many processes arising in the theory of fluid filtration in cracked media are described by discontinuous coefficient pseudoparabolic equations [8-9]. Therewith, the important principal moment is that the considered equation possesses discontinuous coefficients that satisfy only some P -integrability and boundedness conditions, i.e. the considered pseudoparabolic differential operator has no traditional adjoint operator. Therefore, the Riemann function for such

an equation can't be investigated by the classic method of characteristics. It should be specially noted that up to now in references, it was succeeded to construct the Riemann function for different classes of equations only for the case of sufficiently smooth coefficients.

In the present paper, here consider characteristic problem with conditions, setting on all boundary, in two order space for sixth order equation with discontinuous coefficients. The coefficients in this pseudoparabolic equation are not necessarily differentiable; therefore, there does not exist a formally adjoint differential equation making a certain sense. For this reason, this question cannot be investigated by the well-known methods using classical integration by parts and Riemann functions or classical-type fundamental solutions. The theme of the present paper, devoted to the investigation of boundary-value problems for sixth order differential equations of pseudoparabolic type, according to the above-stated is very actual for the solution of theoretical and practical problems. From this point of view, the paper is devoted to the actual problems of mathematical physics and operational research.

2. Problem Statement

Consider the equation

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$$\begin{aligned} (V_{3,3}u)(x,y) &\equiv \sum_{i,j=0}^3 a_{i,j}(x,y) D_x^i D_y^j u(x,y) \\ &= Z_{3,3}(x,y) \in L_p(G), \end{aligned} \quad (1)$$

where $a_{3,3}(x,y) \equiv 1$.

Here $u(x,y)$ is a desired function defined on G ; $a_{i,j}(x,y)$ are the given measurable function on $G = G_1 \times G_2$, where $G_k = (0, h_k)$, $k = 1, 2$; $Z_{3,3}(x,y)$ is a given measurable function on G ; $D_t^k = \partial^k / \partial t^k$ is a generalized differentiation operator in S.L.Sobolev sense; D_t^0 is an identity transformation operator.

Equation (1) is a hyperbolic equation possessing two real characteristics $x = \text{const}$, $y = \text{const}$, the first and second of which is three-fold. Therefore, in some sense we can consider equation (1) as a pseudoparabolic equation [10]. This equation is a generalization of the equation of thin spherical shell bending [11, p. 258].

In this paper we consider equation (1) in the general case when the coefficients $a_{i,j}(x,y)$ are non-smooth functions satisfying only the conditions:

2.1. Problem Statement in Classical Form

We can give the classic form conditions on all the boundary for equation (1) as follows [12]:

$$\begin{cases} u(0,y) = \phi_1(y); u(x,0) = \psi_1(x); \\ u(h_1,y) = \phi_2(y); u(x,h_2) = \psi_2(x); \\ \left. \frac{\partial u(x,y)}{\partial x} \right|_{x=0} = \phi_3(y); \left. \frac{\partial u(x,y)}{\partial y} \right|_{y=0} = \psi_3(x); \end{cases} \quad (2)$$

where $\phi_k(y), \psi_k(x), k = \overline{1,3}$ are the given measurable functions on G . Obviously, in the case of conditions (2) in addition to conditions

$$\phi_k(y) \in W_p^{(3)}(G_2) \equiv \left\{ \tilde{\phi}(y) : D_y^j \tilde{\phi}(y) \in L_p(G_2), j = \overline{0,3} \right\}, \quad 1 \leq p \leq \infty;$$

$$\psi_k(x) \in W_p^{(3)}(G_1) \equiv \left\{ \tilde{\psi}(x) : D_x^j \tilde{\psi}(x) \in L_p(G_1), j = \overline{0,3} \right\}, \quad 1 \leq p \leq \infty,$$

the given functions satisfy also the following agreement conditions (see. Fig. 1):

$$\begin{aligned} a_{i,j}(x,y) &\in L_p(G), \quad i = \overline{0,2}, \quad j = \overline{0,2}; \\ a_{3,j}(x,y) &\in L_{\infty,p}^{x,y}(G), \quad j = \overline{0,2}. \\ a_{i,3}(x,y) &\in L_{p,\infty}^{x,y}(G), \quad i = \overline{0,2}. \end{aligned}$$

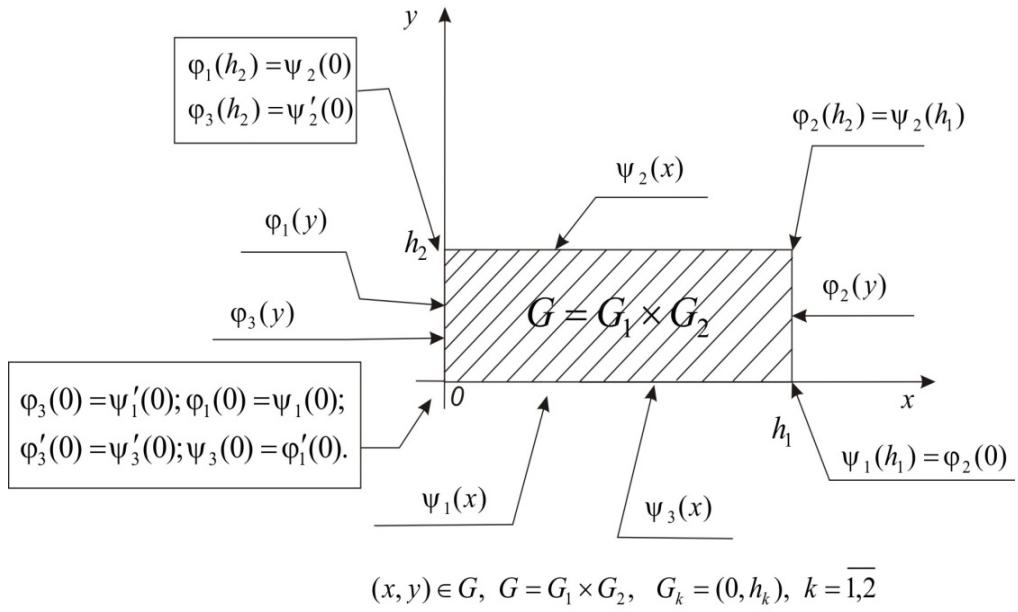
There with the important principal moment is that the equation under consideration possesses discontinuous coefficients that satisfy only some p -integrability and boundedness conditions, i.e. the considered differential operator $V_{3,3}$ has no traditional conjugated operator.

Under this conditions, the solution $u(x,y)$ of equation (1) will be sought in S.L.Sobolev space

$$W_p^{(3,3)}(G) = \left\{ u(x,y) : D_x^i D_y^j u(x,y) \in L_p(G), i = \overline{0,3}, j = \overline{0,3} \right\},$$

where $1 \leq p \leq \infty$. We'll define the norm in the isotropic space $W_p^{(3,3)}(G)$ by the equality

$$\|u\|_{W_p^{(3,3)}(G)} = \sum_{i,j=0}^3 \|D_x^i D_y^j u\|_{L_p(G)}.$$



$$(x, y) \in G, G = G_1 \times G_2, G_k = (0, h_k), k = \overline{1, 2}$$

Figure 1. Geometric interpretation of classical conditions on all the boundary

$$\begin{cases} \varphi_1(h_2) = \psi_2(0); \varphi_3(0) = \psi_1'(0); \\ \varphi_1(0) = \psi_1(0); \varphi_3(h_2) = \psi_2'(0); \\ \psi_1(h_1) = \varphi_2(0); \varphi_3'(0) = \psi_3'(0); \\ \varphi_2(h_2) = \psi_2(h_1); \psi_3(0) = \varphi_1'(0); \psi_3(h_1) = \varphi_2'(0) \end{cases} \quad (3)$$

Obviously, conditions (2) are close to boundary conditions of the Dirichlet problem from [13].

2.2. Problem Statement in Non-classical Treatment

Consider the following non-classical boundary conditions:

$$\begin{cases} V_{0,j}u \equiv D_y^j u(0,0) = Z_{0,j} \in R, j = \overline{0,2}; \\ (V_{0,3}u)(y) \equiv D_y^3 u(0,y) = Z_{0,3}(y) \in L_p(G_2); \\ V_{1,j}u \equiv D_x D_y^j u(0,0) = Z_{1,j} \in R, j = \overline{0,2}; \\ (V_{1,3}u)(y) \equiv D_x D_y^3 u(0,y) = Z_{1,3}(y) \in L_p(G_2); \\ V_{2,j}u \equiv D_x^2 D_y^j u(0,0) = Z_{2,j} \in R, j = \overline{0,1}; \\ (V_{3,j}u)(x) \equiv D_x^3 D_y^j u(x,0) = Z_{3,j}(x) \in L_p(G_1), j = \overline{0,1}; \\ V_{0,j}^{(h_1)}u \equiv D_y^j u(h_1,0) = Z_{0,j}^{(h_1)} \in R, j = \overline{0,2}; \\ (V_{0,3}^{(h_1)}u)(y) \equiv D_y^3 u(h_1,y) = Z_{0,3}^{(h_1)}(y) \in L_p(G_2); \\ V_{i,0}^{(h_2)}u \equiv D_x^i u(0,h_2) = Z_{i,0}^{(h_2)} \in R, i = \overline{0,2}; \\ (V_{3,0}^{(h_2)}u)(x) \equiv D_x^3 u(x,h_2) = Z_{3,0}^{(h_2)}(x) \in L_p(G_1). \end{cases} \quad (4)$$

3. Methodology

The Riemann's method does not work for differential equations with discontinuous coefficients.

In the present paper, a method that essentially uses modern methods of the theory of functions and functional analysis is worked out for investigations of such problems. In the main, this method it requested in conformity to pseudoparabolic equations of sixth order with triple characteristics. Notice that, in this paper the considered equation is a generation of many model equations of some processes (for example, generalized equations of moisture transfer, telegraph equations, string vibrations equations and etc).

3.1. Method Agreement Conditions

If the function $u(x, y) \in W_p^{(3,3)}(G)$ is a solution of the classical form problem (1), (2), with conditions on all the boundary then it is also a solution of problem (1),(4) for $Z_{i,j}$, $Z_{i,j}^{(h_1)}$, $Z_{i,j}^{(h_2)}$, determined by the following equalities:

$$Z_{0,0} = \varphi_1(0) = \psi_1(0);$$

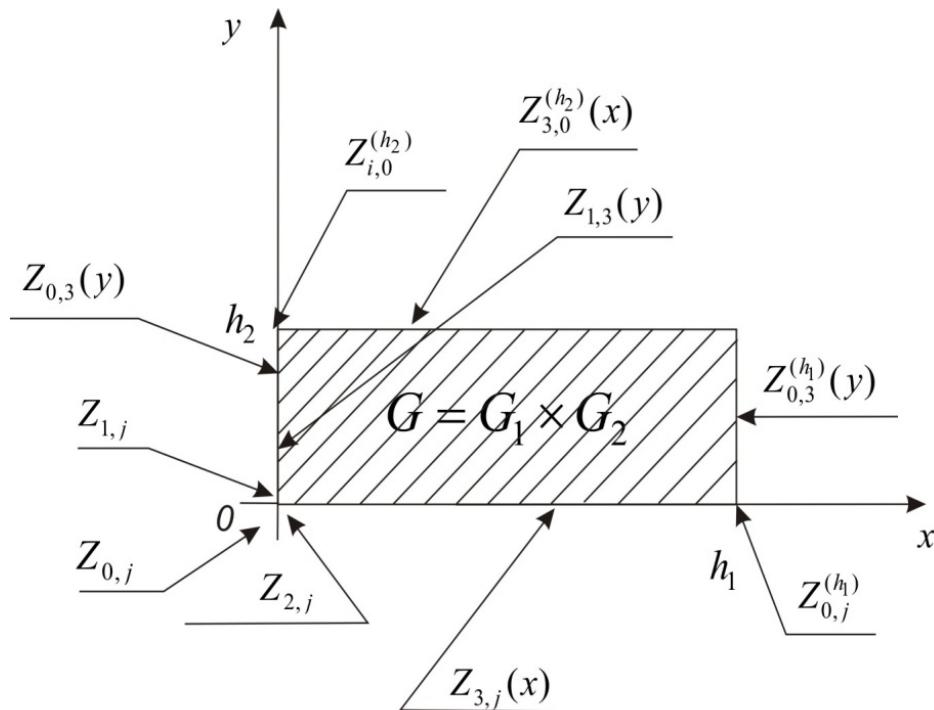
$$Z_{0,1} = \varphi_1'(0) = \psi_3(0); \quad Z_{0,2} = \varphi_1''(0);$$

$$Z_{0,3}(y) = \varphi_1'''(y); \quad Z_{1,0} = \varphi_3(0) = \psi_1'(0);$$

$$\begin{aligned} Z_{1,1} &= \varphi_3'(0) = \psi_3'(0); \quad Z_{1,2} = \varphi_3''(0); \\ Z_{1,3}(y) &= \varphi_3'''(y); \quad Z_{2,0} = \psi_1''(0); \\ Z_{3,0}(x) &= \psi_1'''(x); \quad Z_{2,1} = \psi_3''(0); \\ Z_{3,1}(x) &= \psi_3'''(x); \quad Z_{0,0}^{(h_1)} = \varphi_2(0) = \psi_1(h_1); \\ Z_{0,1}^{(h_1)} &= \varphi_2'(0) = \psi_3(h_1); \quad Z_{0,2}^{(h_1)} = \varphi_2''(0); \\ Z_{0,3}^{(h_1)}(y) &= \varphi_2'''(y); \\ Z_{0,0}^{(h_2)} &= \psi_2(0) = \varphi_1(h_2); \\ Z_{1,0}^{(h_2)} &= \psi_2'(0) = \varphi_3(h_2); \quad Z_{2,0}^{(h_2)} = \psi_2''(0); \\ Z_{3,0}^{(h_2)}(x) &= \psi_2'''(x). \end{aligned}$$

3.2. Method Integral Representation

It is easy to prove that the inverse one is also true. In other words, if the function $u \in W_p^{(3,3)}(G)$ is a solution of problem (1), (4) (see. Fig. 2), then it is also a solution of problem (1), (2) for the following functions:



$$(x, y) \in G, \quad G = G_1 \times G_2, \quad G_k = (0, h_k), \quad k = \overline{1, 2}$$

Figure 2. Geometrical interpretation of conditions on all the boundary in non-classical treatment

$$\varphi_1(y) = Z_{0,0} + yZ_{0,1} + \frac{y^2}{2}Z_{0,2} + \frac{1}{2} \int_0^y (y-\tau)^2 Z_{0,3}(\tau) d\tau; \quad (5)$$

$$\varphi_2(y) = Z_{0,0}^{(h_1)} + yZ_{0,1}^{(h_1)} + \frac{y^2}{2}Z_{0,2}^{(h_1)} + \frac{1}{2} \int_0^y (y-\xi)^2 Z_{0,3}^{(h_1)}(\xi) d\xi; \quad (6)$$

$$\varphi_3(y) = Z_{1,0} + yZ_{1,1} + \frac{y^2}{2}Z_{1,2} + \frac{1}{2} \int_0^y (y-\eta)^2 Z_{1,3}(\eta) d\eta; \quad (7)$$

$$\psi_1(x) = Z_{0,0} + xZ_{1,0} + \frac{x^2}{2}Z_{2,0} + \frac{1}{2} \int_0^x (x-\tau)^2 Z_{3,0}(\tau) d\tau; \quad (8)$$

$$\psi_2(x) = Z_{0,0}^{(h_2)} + xZ_{1,0}^{(h_2)} + \frac{x^2}{2}Z_{2,0}^{(h_2)} + \frac{1}{2} \int_0^x (x-\nu)^2 Z_{3,0}^{(h_2)}(\nu) d\nu; \quad (9)$$

$$\psi_3(x) = Z_{0,1} + xZ_{1,1} + \frac{x^2}{2}Z_{2,1} + \frac{1}{2} \int_0^x (x-\mu)^2 Z_{3,1}(\mu) d\mu; \quad (10)$$

Note that the functions (5)-(10) possess one important property, more exactly, agreement condition (3) for all $Z_{i,j}$, $Z_{i,j}^{(h_1)}$, $Z_{i,j}^{(h_2)}$ having the above-stated properties are fulfilled for them automatically. Therefore equalities (5)-(10) may be considered as a general form of all the functions $\varphi_k(y)$, $\psi_k(x)$, $k = \overline{1,3}$, satisfying agreement conditions (3).

4. Result

So, classic type problems (1), (2) and of the form (1), (4) with conditions on all the boundary are equivalent in general case. However, the non - classic problem (1), (4) with conditions on the boundary is more natural by statement than problem (1), (2).

This is connected with the fact that in the statement of problem (1), (4) with conditions on all the boundary, the right parts of boundary conditions don't require additional conditions of agreement type. Note that various boundary value problems with non classic conditions not requiring agreement conditions were substantiated in the author's papers [14-15].

5. Discussion and Conclusions

In the paper a non-classical type boundary problem is substantiated for a pseudoparabolic equation with non-smooth coefficients and with a sixth order dominating derivative. Classic boundary conditions are reduced to

non-classic conditions by means of integral representations. Such statement of the problem has several advantages:

- 1) No additional agreement conditions are required in this statement;
- 2) One can consider this statement as a problem formulated in terms of traces in the Sobolev space $W_p^{(3,3)}(G)$;
- 3) In this statement the considered equation is a generalization of many model equations of some processes (e.g. moisture transfer equation, telegraph equation, string vibration equations and etc.).

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