Left Quasi-Artinian Modules

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Abstract In this paper we study a new class of left quasi-Artinian modules. We show: If \( R \) is a left quasi-Artinian ring and \( M \) is a left \( R \)-module, then (a) \( \text{Soc}(M) \) ess \( M \) and (b) \( \text{Rad}(M) \) small in \( M \). Then we prove: if \( I \) is a non-nilpotent left ideal in a left quasi-Artinian ring, then \( I \) contains a non-zero idempotent element. Finally, we show that a commutative ring \( R \) is quasi-Artinian if and only if \( R \) is a direct sum of an Artinian ring with identity and a nilpotent ring.

Keywords Modules with Chain Conditions, Left Quasi-Artinian Modules and Nilpotent Rings

1. Introduction

By ring we mean an associative ring that need not have an identity. In this paper, we study a new class of left quasi-Artinian Modules, which is a generalization of left Artinian modules. First we study the problems of finding conditions which are equivalent to the definition of left quasi-Artinian Module (Theorem 1.2). Then we show that the class of left quasi-Artinian Modules is Q-closed, S-closed and E-closed.

In section two we study the module structures over left quasi-Artinian ring, in particular we prove that if \( R \) is a left quasi-Artinian ring, then every finitely generated left \( R \)-module \( M \) is a left quasi-Artinian (Theorem 2.1). Finally we show that: if \( R \) be a ring, \( N = N(R) \), then \( R \) is a left quasi-Artinian if and only if \( N \) is nilpotent and each of the \( R/N, N/N^2, N^2/N^3, \ldots \) is left quasi-Artinian \( R \)-module (Theorem 2.4).

In section three we describe the ideal structures and we give some classification, in particular we prove that if \( I \) is a non-nilpotent left ideal in a left quasi-Artinian ring, then \( I \) contains a non-zero idempotent element (Theorem 3.2). Next we prove that if \( R \) is a semi-prime left quasi-Artinian ring and \( I \) be a non-zero left ideal of \( R \), then \( I = Re \) for some non-zero idempotent \( e \) in \( R \) (Theorem 3.5).

1.1. Definitions and Basic Properties

Let \( M \) be a left \( R \)-module. We say that \( M \) is a left quasi-Artinian Module if for every descending chain \( N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots \) of left \( R \)-submodules of \( M \), there exist \( m \in \mathbb{Z}^+ \) such that \( R^m N_m \subseteq N_n \) for all \( n \).

It is clear that any left Artinian module is left quasi-Artinian and it is easy to prove the following

Lemma 1.1

Let \( M \) be a left \( R \)-module.

(a) If \( RM = 0 \), then \( M \) is a left quasi-Artinian.

(b) If \( R \) has an identity and \( M \) is unitary, then \( M \) is left quasi-Artinian if and only if \( M \) is left Artinian.

Now we prove the following which is a characterization of left quasi-Artinian modules.

Theorem 1.2

Let \( M \) be a left \( R \)-module. Then the following conditions are equivalent:

\( \zeta \) of left \( R \)-submodules of \( M \) such that (a) In every non-empty collection \( K \in \zeta \), then \( RK \in \zeta \), there exists a minimal element.

that if

(b) For every descending chain of left \( R \)-submodules \( N_1 \supseteq N_2 \supseteq \ldots \)

\( R^m N_1 \supseteq R^m N_2 \supseteq \ldots \) there exists \( m \in \mathbb{Z}^+ \) such that a descending chain terminates.

(c) \( M \) is left quasi-Artinian.

(d) For every non-empty collection \( \zeta \) of left \( R \)-submodules of \( M \), there exists \( N \in \zeta \) and \( m \in \mathbb{Z}^+ \) such that \( R^m N \subseteq K \) for any \( K \in \zeta \), \( K \subseteq N \).

Proof:

(a \( \Rightarrow \) b) Suppose that \( N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots \) is a descending chain of left \( R \)-submodules of \( M \) but the descending chain \( R^m N_1 \supseteq R^m N_2 \supseteq \ldots \supseteq R^m N_n \supseteq \ldots \) of left \( R \)-submodules of \( M \) does not terminate for all \( m \in \mathbb{Z}^+ \).

Therefore the collection \( \zeta = \{ N_1, N_2, \ldots, RN_1, RN_2, \ldots, R^m N_1, R^m N_2, \ldots \} \) is a nonempty collection of \( R \)-submodules and for all \( N \in \zeta \) we have \( RN \in \zeta \). Hence \( \zeta \) has no minimal element, which
is a contradiction. 

(b) ⇒ (c) Let \( N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots \) be any descending chain of left \( R \)-submodules of \( M \) then there exists \( m \in \mathbb{Z}^+ \) such that \( R^m N_1 \supseteq R^m N_2 \supseteq \ldots \supseteq R^m N_n \supseteq \ldots \) form a descending chain of left \( R \)-submodules of \( M \) and by (b) there exists \( s \in \mathbb{Z}^+ \) such that \( R^m N_s = R^m N_n \) for all \( n \geq s \), but \( R^m N_s \subseteq N_n \) for all \( n \geq s \). Take \( t = \max \{m, s\} \) then \( R^t N_t \subseteq N_n \) for all \( n \), hence \( M \) is a left quasi-\( \text{Artinian} \).

(c) ⇒ (d) Let \( \zeta \) be a non-empty collection of left \( R \)-submodules of \( M \) such that for each \( N \in \zeta \) and \( m \in \mathbb{Z}^+ \), there exists \( K \in \zeta \) such that \( K \subseteq N \), but \( R^m N \not\subseteq K \). Now let \( N_1 \in \zeta \) then there exists \( N_2 \in \zeta \) such that \( R N_1 \not\subseteq N_2 \), where \( N_1 \supseteq N_2 \), but \( N_2 \in \zeta \), hence there exists \( N_3 \in \zeta \), such that \( R^2 N_2 \not\subseteq N_3 \). Continuing in this manner we can construct an infinite descending chain \( N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots \) of left \( R \)-submodules of \( M \) such that \( R^m N_m \not\subseteq N_{m+1} \), \( m=1,2,\ldots \). Hence \( R^m N_m \not\subseteq N_n \) for some \( n \), which is a contradiction.

(d) ⇒ (a) Let \( \zeta \) be a non-empty collection of left \( R \)-submodules of \( M \) such that \( RK \subseteq \zeta \) for all \( K \in \zeta \). Then \( R^m K \subseteq \zeta \), for all \( m \in \mathbb{Z}^+ \). But \( R^m K \subseteq \zeta \) for all \( m \in \mathbb{Z}^+ \), hence by (d) there exists an \( s \in \mathbb{Z}^+ \) such that \( R^s K \subseteq \zeta \). Therefore if \( m \geq s \), then \( R^s K = R^m K \) and \( \zeta \) has a minimal element.

Next we prove the following:

**Proposition 1.3**

Let \( M \) be a left \( R \)-module. If \( RM \) is left \( \text{Artinian} \), then \( M \) is left quasi-\( \text{Artinian} \).

**Proof:**

be a descending chain of left \( R \)-submodules of \( M \). Let \( N_1 \supseteq N_2 \supseteq \ldots \)

\( R \)-submodules of \( M \) such that \( R N_1 \supseteq R N_2 \supseteq \ldots \) is a descending chain of then \( R M \). But \( RM \) is left \( \text{Artinian} \), hence there exists \( s \in \mathbb{Z}^+ \) such that \( R N_s = R N_n \). Further \( R^s N_s \subseteq R N_n \). For all \( n \) Hence \( M \) is left quasi-\( \text{Artinian} \).

**Remark:** The converse of Proposition 1.3 needs not be true as the following example shows:

Let \( M = \begin{bmatrix} Q & 0 \\ Q & 0 \end{bmatrix} \) and \( R = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \). Then \( M \) is left quasi-\( \text{Artinian} \).

\[ R \text{-module, but } RM = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} = R \text{ is not left \( \text{Artinian} \).} \]

Now let \( \mathcal{M} \) be a class of modules. Then we say that \( \mathcal{M} \) is \( S \)-closed if \( N \) is a submodule of \( M \) and \( M \in \mathcal{M} \), then \( N \in \mathcal{M} \). We say that \( \mathcal{M} \) is \( Q \)-closed if \( M \in \mathcal{M} \) and \( N \) is a submodule of \( M \), then \( M/N \in \mathcal{M} \). We say that \( \mathcal{M} \) is \( E \)-closed if \( N \) is a submodule of \( M \) and \( N, M/N \in \mathcal{M} \), then \( M \in \mathcal{M} \).

**Proposition 1.4**

Let \( \mathcal{M} \) be the class of left quasi-\( \text{Artinian} \) modules. Then (a) \( \mathcal{M} \) is \( S \)-closed. (b) \( \mathcal{M} \) is \( Q \)-closed. (c) \( \mathcal{M} \) is \( E \)-closed.

**Proof:**

(a) is clear

(b) Suppose that \( M \) is a left quasi-\( \text{Artinian} \) \( R \)-module and \( N \) is submodule of \( M \). Let \( \pi: M \to M/N = \overline{M} \) be the natural homorphism of left quasi-\( \text{Artinian} \) module onto \( \overline{M} \). Then \( N \supseteq N_2 \supseteq \ldots \) is a descending chain of submodules of \( \overline{M} \), and \( N \supseteq N_2 \supseteq \ldots \) is a descending chain of \( R \)-submodules of \( M \), where \( N_i = \pi^{-1}(\overline{N}_i) \) but \( M \) is left quasi-\( \text{Artinian} \), hence there exists \( m \in \mathbb{Z}^+ \) such that \( R^m N_m \subseteq N_n \) for all \( n \). But \( (N_k) = \overline{N}_k \). Hence \( R^m N_m \subseteq \overline{N}_n \) for all \( n \). Therefore \( \overline{M} \) is left quasi-\( \text{Artinian} \).

(c) Suppose that \( N \) be a \( R \)-submodule of \( M \) and \( N, M/N \in \mathcal{M} \). Let \( \mathcal{M} \) be a descending chain of left \( R \)-submodules of \( M \). Then \( N_1 \supseteq N_2 \supseteq \ldots \)

\( N_1 \cap N \supseteq N_2 \cap N \supseteq \ldots \) is a descending chain of \( R \)-submodules of \( N \). But \( s \in \mathbb{Z}^+ \) such that left quasi-\( \text{Artinian} \), hence there exists \( N \)

\[ N^s(N_s \cap N) \subseteq N_n \cap N \] for all \( n \). Now \( N_1 + N/N \supseteq N_2 + N/N \supseteq \ldots \) is a descending chain of submodules of \( M/N \) and \( N_1 + N/N \) is left quasi-\( \text{Artinian} \), therefore there exists \( k \) such that \( R^k(N_k + N/N) \subseteq N_n + N/N \) for all \( n \). That is \( R^k(N_k + N/N) \subseteq N_n + N/N \) for all \( n \). Now let \( m = \max\{s, k\} \) Then \( R^m (N_m + N/N) \subseteq N_n \cap N \) and \( R^m (N_m + N/N) \subseteq N_n \cap N \) for all \( n \).

Now \( R^m N_m = R^m [N_m \cap (N_m + N/N)] \)

\[ \subseteq [N_m \cap (N_m + N/N)] \] by modular law, therefore \( R^m N_m = N_m \cap (N_m + N/N) \) for all \( n \). Hence \( R^m N_m \subseteq R^m N_m \subseteq N_m \) for all \( n \). Therefore \( M \) is left quasi-\( \text{Artinian} \). An immediate consequence of Proposition 1.4, we have the following:

**Corollary**

Let \( \mathcal{M} \) be the class of quasi-\( \text{Artinian} \) modules. If \( M = A + B \)
where A, B in \( \mathfrak{M} \) then \( M \in \mathfrak{M} \).

Remark: Suppose that \( R \) has \( 1, \) so \( M = M_1 \oplus M_2 \) where
\[
M_1 = \{ lm : m \in M \} \quad \text{and} \quad M_2 = \{ m - lm : m \in M \}.
\]
Here \( M_1 \) is unitary and left quasi-Artinian if and only if \( M \) is left \( \mathfrak{M} \). \( R M_2 = 0 \) quasi-Artinian if and only if \( M_1 \) is left Artinian. And \( M \) is left Artinian if and only if \( M_1 \) and \( M_2 \).

2. The Submodule Structures

In this section we study the submodules structure by consider modules over left quasi-Artinian ring. First we prove the following

**Theorem 2.1**

Let \( R \) be a left quasi-Artinian ring. Then every finitely generated left \( R \)-module is quasi-Artinian.

**Proof:**

Let \( L \) be a finitely generated left \( R \)-module, then \( L = R x_1 + R x_2 + \cdots + R x_n \) where \( 0 \neq x_i \in M, \ 1 \leq i \leq n \). If \( n = 1 \) then \( M \) is cyclic and therefore isomorphic to \( R/L \) where
\[
L = \{ a \in R \mid ax_1 = 0 \}.
\]
Since \( R/L \) is left quasi-Artinian, so is every factor module. Assume inductively that the Theorem holds for modules which can be generated by \( n - 1 \) or fewer elements. Then \( Rx_1 \) is left quasi-Artinian and
\[
M/Rx_1 \cong (Rx_1 + Rx_2 + \cdots + Rx_n)/Rx_1 \cong (Rx_2 + \cdots + Rx_n)/Rx_1 \cap (Rx_2 + \cdots + Rx_2)
\]
which is left quasi-Artinian. Therefore \( M \) is left quasi-Artinian. Let \( R \) be a ring and \( M \) is a left \( R \)-module. Then
\[
(a) \ Soc(M) = \sum \{ K \leq M : K \text{ is simple in } M \}
\]
\[
= \cap \{ L \leq M : L \text{ is essential in } M \}
\]
\[
(b) \ Rad(M) = \cap \{ K : K \text{ is maximal submodule in } M \}
\]
\[
= \sum \{ L : L \text{ is small submodule in } M \}
\]

**Theorem 2.2**

Let \( R \) be a left quasi-Artinian ring and \( M \) is a left \( R \)-module. Then

(a) \ SocM ess \( M \)

(b) \ RadM s max \( M \)

**Proof:**

(a) \( \rho_x : R \rightarrow Rx \) such that \( \rho_x (r) = rx \) for \( r \in R \) is a homomorphism of \( R \) onto the submodule \( Rx \) with
\[
\text{Kernel } \ker \rho_x = \{ r \in R \mid rx = 0 \}.
\]
So \( R/\ker \rho_x \cong Rx \). But \( R \) is left quasi-Artinian, hence by Proposition 1.4, \( Rx \) is left quasi-Artinian. We claim that \( Rx \) contains a minimal submodule. To prove this let \( l = \{ N \subseteq Rx \mid 0 \neq x \in M, N \leq M \} \) be a nonempty collection of \( R \)-submodule of \( Rx \) and \( J \in l \) then \( J = Ry \) for some \( 0 \neq y \in M \). But \( J/\ker \rho_y = (\ker \rho_y) y = R^2 y \subseteq Ry = J \in l \). But \( l \) has a minimal element, hence \( Soc(R) \neq 0 \). But \( Soc(Rx) = Rx \cap Soc(M) \), hence \( Soc(M) \) is a \( \mathfrak{M} \).

(b) First we show that \( Rad(M) = J M \) where \( J = J(R) \).

Since for any left \( R \)-module \( M \) the factor module \( R/M \) is simple, \( J = \{ m \in M \mid \text{ for any } \lambda \in \mathbb{C}, \lambda m = 0 \} \). Therefore \( J(R) \) is annihilator of every simple \( R \)-submodule of \( M \), hence \( J(R) \) is left Artinian if and only if \( J(R) \) is semi-simple.

Conversely since \( J(R) \) is semi-simple then we have \( Soc(M) = R \). Therefore \( Soc(M/JM) = R/M \) (J(R)/J) = R/M (0) = M/JM \). Hence \( M/JM \) is semi-simple. By Theorem 2.2, since \( Rad(M) \) is small in \( M \), then \( M/JM \) is semi-simple.

**Corollary 2.3**

Let \( R \) be left quasi-Artinian ring and \( M \) left \( R \)-module, then \( M \) is finitely generated if and only if \( M/Rad(M) \) is finitely generated.

**Proof:**

By Theorem 2.2, since \( Rad(M) \) is small in \( M \), then the result follows.

By the nil radical \( N = N(R) \) of a ring \( R \) we mean the sum of all nilpotent ideals of \( R \), which is a nil ideal. It is well known [7, P.28 Theorem 2], that \( N \) is the sum of all nilpotent left ideals of \( R \) and it is the sum of all nilpotent right ideals of \( R \).

Now we give another characterization of left quasi-Artinian ring, namely the following:

**Theorem 2.4**

Let \( R \) be a ring. \( N = N(R) \) be the nil radical of \( R \), then \( R \) is a left quasi-
\[ N \]

(n) nilpotent and each of \( R/N, N/N^2 , N^2/N^3 \), ..., Artinian if and only if it is left quasi-Artinian.

**Proof:**

Suppose \( R \) is left quasi-Artinian. Then by [3, Corollary 2.3] \( N \) is nilpotent. Now let \( R = L \). Then \( M \) is left quasi-Artinian. \( R \)-module and \( N^i \) is an ideal of \( R \) for all \( i \). Therefore \( N^i \) is a \( R \)-submodule of \( M \) for all \( i \). By Proposition 1.4, \( R/N^i \) is left quasi-Artinian for all \( i \). Also \( N^i/N^{i+1} \) is a \( R \)-submodule of \( R/N^{i+1} \) so each \( N^{i+1}/N^{i+1} \) is left quasi-Artinian.

To prove the converse, note that since \( R/N \cong R/N^2 \), if it follows from Proposition 1.4, that \( R/N^2 \) is left quasi-Artinian \( R \)-module and by induction \( R/N \) is left quasi-Artinian for all \( i \). But \( N \) is nilpotent, hence there exists \( m \in \mathbb{Z}^+ \) such that \( N^m = 0 \), therefore \( R \cong R/N^m \) is left quasi-Artinian \( R \)-module.
Hence $R$ is left quasi-Artinian ring.

3. The Ideal Structures

In this section we study the ideal structures in a left quasi-Artinian ring. Note that if $R=I=R_1+R_2$ where $R_1,R_2$ are left quasi-Artinian. Let $I_1 \supseteq I_2 \supseteq \ldots$ be a descending chain of left ideals of $R$. Then $R_1I_1 \supseteq R_1I_2 \supseteq \ldots$ is a descending chain of left ideals of $R_1$ and $R_2I_1 \supseteq R_2I_2 \supseteq \ldots$ is a descending chain of left ideals of $R_2$ but $R_1,R_2$ are left quasi-Artinian rings, hence there exist $r,s$ such that $R_1^r(R_1I_1) \subseteq R_1I_n \subseteq I_n$ and $R_2^s(R_2I_1) \subseteq R_2I_n \subseteq I_n$. Let $m=\max\{r,s\}$. Then $R_1^m(R_1I_1) \subseteq R_1I_n \subseteq I_n$ and $R_2^m(R_2I_1) \subseteq R_2I_n \subseteq I_n$ for all $n$. But $R_1=\bigoplus_{i=1}^{m} R_1^i \oplus R_1^{m+1}$ and $R_2=\bigoplus_{i=1}^{m} R_2^i \oplus R_2^{m+1}$, hence $R^{m+1}I_1 = R_1^m(R_1I_1) + R_2^m(R_2I_1) \subseteq I_n$ for all $n$ and $R^{n+1}I_1 \subseteq R^{n+1}I_n \subseteq I_n$ for all $n$. Therefore $R$ is a left quasi-Artinian.

Theorem 3.2

Let $I$ be a non-nilpotent left ideal in a left quasi-Artinian ring, then $I$ contains a non-zero idempotent element.

To prove this we need the following lemma.

Lemma 3.3

Let $R$ be a left quasi-Artinian ring. Then every non-nilpotent left ideal of $R$ contains a minimal non-nilpotent left ideal.

Proof:
Let $I$ be a non-nilpotent left ideal of $R$ and suppose that $I$ does not contain a minimal non-nilpotent left ideal of $R$. Then $0 \neq I^2 \subseteq RI \subseteq I$ and $RI$ is not nilpotent. Therefore there exists a non-nilpotent left ideal $I_1 \nsubseteq RI \subseteq I$. Hence $0 \neq I_1^3 \subseteq R^2I_1$ and $R^2I_1$ is not nilpotent. In this way we can find a non-nilpotent left ideal $I_n \nsubseteq R^{n-1}I_{n-1} \subseteq I_{n-1}$ then $0 \neq I_{n+1} \subseteq R^nI_n$ and $R^nI_n$ is not nilpotent and so on. Hence $I \supseteq I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ is an infinite descending chain of left ideals of $R$ which is a contradiction. Therefore $I$ contains a minimal non-nilpotent left ideal of $R$.

Proof of Theorem

Let $I$ be non-nilpotent non-nilpotent left ideal of $R$. Since $R$ is a left quasi-Artinian ring, then by Lemma 3.3, $I$ contains a minimal non-nilpotent left ideal $K$. Since $K^2 \neq 0$ then there exists $x \in K$ such that $xK \neq 0$. However $xK \subseteq K$ and $xK$ is a left ideal of $R$, hence by minimality of $K$ we have $xK=K$. Therefore there exists $e \in K$ such that $xe=x$ and since $xe^2=xe$ we get that $x(e^2-e)=0$.

Now, let $K_0=\{a \in K \mid xa=0\}$, therefore $K_0$ is a left ideal of $R$ and $K_0 \subseteq K$ since, $xK \neq 0$, for all $x \in K$.

Therefore we must have $K_0=0$ and $(e^2-e) \in K_0$. Hence $e^2=e$. Since $xe=x \neq 0$ we have that $e \neq 0$. Now, $Re \subseteq K$ is a left ideal of $R$ and contains $e^2=e \neq 0$.

so that $Re \neq 0$, then $e \in Re=K \subseteq I$. Hence $e \in I$.

Corollary 3.4

If $R$ is left quasi-Artinian ring, then every nil left ideal of $R$ is a nilpotent.

Proof:
Let $N$ be a non-zero nil left ideal of $R$ and suppose that $N$ is not nilpotent. Then by Theorem 3.2, there exists a nonzero idempotent element $e$ and $e \in N$. Therefore $e$ is nilpotent which is a contradiction. Hence $N$ must be nilpotent.

Next we prove the following theorem.

Theorem 3.5

Let $R$ be a semi-prime left quasi-Artinian ring and $I$ be a nonzero left ideal of $R$, then $I=Re$ for some nonzero idempotent $e$ in $R$.

Proof:
Since $I$ is not nilpotent, it follows from Theorem 3.2, that $I$ contains a non-zero idempotent element say, $e$. Let $A(e)=\{x \in I \mid xe=0\}$ then the set of left ideals $L=\{A(e) \mid 0 \neq e^2=e \in I\}$ is not empty. Now, if $A(e) \subseteq L$, then $RA(e) \subseteq L$. Now since $I$ is a left ideal of $R$, then $re \in I$, where $r \in R$, $e \in I$, therefore $0 \neq re^2=re \in I$. But $R$ is a left quasi-Artinian, hence by Theorem 1.2, $R$ has a minimal element $A(e_0)$, say. Either $A(e_0) \neq 0$ or $A(e_0)=0$. If $A(e_0) \neq 0$, then $A(e_0)$ must have an idempotent $e_1$, say. By definition of $A(e_0)$, $e_1 \in I$. Consider $e_2=e_0+e_1-e_0e_1$, then
Theorem $R/N$ is a finite direct sum of its minimal ideals, each of which is a simple Artinian ring, that is
\[ R/N \cong \bigoplus_{i=1}^{n} \bar{N}_i, \]
where $\bar{N}_i = \langle \bar{e}_i \rangle$ is a minimal ideal of $R/N$
which is a simple Artinian ring. But a finite direct sum of Artinian is again Artinian, hence $\bigoplus_{i=1}^{n} \bar{N}_i$ is an Artinian ring and $R/N$ is a semi-simple Artinian. But $\bar{N}_i$ is a semi-simple Artinian so, it has an identity element. Therefore $\bigoplus_{i=1}^{n} \bar{N}_i$ is an Artinian ring with identity. Hence,
\[ R \cong \bigoplus_{i=1}^{n} \bar{N}_i \oplus N \]
and $R$ is a direct sum of Artinian ring with identity and nilpotent ring.

Finally we prove the following which characterizes the prime ideals in left Quasi-Artinian rings.

**Theorem 3.8**

Let $R$ be a commutative quasi-Artinian ring and $I$ be a minimal ideal in $R$. Then $ann(I)$ is a maximal ideal.

To prove this we need the following

**Lemma 3.9**

If $R$ is a commutative quasi-Artinian ring and every prime ideal of $R$ is maximal.

Proof:

Let $P$ be a prime ideal of $R$, then $R/P$ is a prime ring. Now $R/P$ is a semi-prime quasi-Artinian ring. Therefore by Corollary 3.5 $R/P$ is a semi-simple Artinian. Hence by Wedderburn’s Theorem $R/P$ is a finite direct sum of minimal ideals, each of which is a simple Artinian ring. But a prime ring cannot be written as a direct sum of non-trivial ideals, hence $R/P$ is a simple ring. Therefore $P$ is maximal.

An immediate consequence of Lemma 3.9 we have the following

**Corollary 3.10**

If $R$ is a quasi-Artinian ring, then $J(R) = \text{rad}(R) = N(R)$, where $J(R)$ is the Jacobson radical of $R$ and $\text{rad}(R)$ is the prime radical of $R$.

Proof of Theorem 3.8

By Lemma 3.10, it is enough to show that $\text{ann}(I)$ is a prime ideal in $R$.

Let $x, y \in R$ such that $x, y \notin \text{ann}(I)$. Then $xI \neq 0$ and $yI \neq 0$, but $xI \subseteq I$ and $yI \subseteq I$. But $I$ is a minimal ideal of $R$, hence $xI = I$ and $yI = I$. Therefore $0 \neq xy \in I$ and $xy \notin \text{ann}(I)$, and $\text{ann}(I)$ is a prime ideal of $R$. 

Corollary 3.6

Any semi-prime left quasi-Artinian ring is a semi-simple left Artinian.

Proof:

By Theorem 3.5 every non-zero left ideal of $R$ is generated by a non-zero idempotent $e$, say. But we know that $e$ acts as right identity for the left ideal $I = Re$, and since $R$ is itself an ideal, hence $R$ has an identity element. Therefore $R$ is left Artinian. Now, $J(R)$ is nilpotent, and $R$ is a semi-prime ring, implies that $J(R) = 0$. Hence $R$ is a semi-simple.

Now we describe left quasi-Artinian rings using the non-commutative version of Wedderburn Theorem. In particular we prove the following

**Theorem 3.7**

A commutative ring $R$ is quasi-Artinian if and only if $R$ is a direct sum of an Artinian ring with identity and a nilpotent ring.

To prove this we need the following

**Lemma 3.8**

Let $R$ be a left quasi-Artinian ring and $N$ be the nil radical of $R$. Then $R/N$ is a semi-simple Artinian ring.

Proof:

Since $N$ is nilpotent and $R/N$ is left quasi-Artinian, it follows that $R/N$ is a semi-prime left quasi-Artinian. Therefore by Corollary 3.5, $R/N$ is a semi-simple Artinian ring.

Proof of Lemma 3.8

Suppose that $R$ is a direct sum of an Artinian ring with identity and a nilpotent ring, since any Artinian ring and any nilpotent ring are quasi-Artinian, it follows that $R$ is a quasi Artinian ring.

To prove the converse. Let $N = N(R)$ be a nil radical of $R$. Then by Corollary 3.4 $N$ is nilpotent and by Lemma 3.8, $R/N$ is a semi-simple Artinian ring. Therefore by Wedderburn's
REFERENCES


