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On Competing Risk Model Under Random Masking

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Abstract Analysts are often interested in obtaining component reliabilities by analyzing system life data. Making a series system assumption and applying a competing risk model generally does this. In practice, the data consists of a system life-length and a subset of components that contains the true cause of system failure (the true cause of system failure is masked from our knowledge). We consider a series system composed of independent Weibull components with common shape parameter and different scale parameters. It is shown that the IFRA-ness of the system is measured by the common shape parameter, which is also known as the aging factor. We derive the consistency and asymptotic normality of the MLEs of the scale parameters under random masking.

Keywords Masked Data, Competing Risk, Weibull Distribution, Maximum Likelihood Estimates, Aging, Schur-Concavity

1. Introduction

In industrial problems, we are often interested in estimating component reliabilities from system lifetime data. In this context, the competing risk model is normally used. That is, component reliability from system life-data is generally estimated by assuming a series system and applying a competing risk model, where the system life-data and also the cause of system failure are known. The combined data can then be employed to estimate life distribution of the system and each of its components. In practice, however, this type of analysis is confounded with a problem of masking (the exact cause of failure is unknown). It is mentioned in[4] that "… investigation of cause of failure is expensive and time consuming, and hence sometimes the cause of failure is not observed, even if the failure time is recorded". When a large computer system fails in the field, analysis is usually performed (see[7]) such that a small subset of components, perhaps a circuit card, is identified as the cause of failure. In an attempt to repair the system as quickly as possible, the entire subset of components is replaced and exact failing component may be investigated further. We call the data, for which cause of failure is narrowed to a subset of components, masked because the true cause of failure is masked from our knowledge. The resulting quantities obtained are then the system-life and the partial information on the cause of failure.[4] considered a model involving a series system of two components in which n systems are put on test and only for m (<n) systems, the failure times and the cause of failure were noted while the remaining n-m observations include only the failure times (i.e., n-m is the number of masked data).

The model in[4] was extended by[7] and[2] to a model where they allowed an arbitrary number of components in series and random masking. They assumed the lifetime of the i-th component to be exponential with unknown but constant parameter λ_i. They obtained the closed form expression for the MLEs of the parameters when the system has two components. For the case of three component system, they used the Picard's method to obtain approximate solution to the MLE of λ_i, i=1,2,3. In[3], the general multi-component system is discussed and they proved consistency of the MLEs under certain conditions.

No study has been reported so far for the above model taking the aging of the component into consideration. Aging is an important aspect in reliability to deal with. We consider a series system consisting of two components having Weibull

$W(\alpha,\beta_j)$ with distribution function

$$F(t;\theta_j)=1-\exp(-\left(\frac{t}{\beta_j}\right)^{\alpha}),t>0.$$

where $\theta_j=(\beta_j,\alpha)$. Let $f(t;\theta_j)$ denote the density of $W(\alpha,\beta_j)$.

The paper is organized as follows. In Section 2, we show that the IFRA (increasing failure rate average)-ness of the series system consisting of Weibul components with common shape parameter α is measured by α. In Section 3, we develop the likelihood function for a n-component series system. In Section 4, we establish the

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consistency and asymptotic normality of the estimates of the scale parameters $\beta_j, j = 1,2.$

2. Notion of Aging

The following definition is due to[1].

Definition 2.1: A system with exchangeable units {1,2…,n} age or wear out with respect to age, if their joint survival function

$$P(X_1 > y_1,..., X_n > y_n) = \overline{F}(y_1,..., y_n)$$

is Schur concave.

Theorem 2.1 (see[1]): Let $X_1,..., X_n$ be the independent lifetimes of n units, each with specified univariate distribution F. Suppose further that $\ln \overline{F}(x)$ is concave when finite. Then

$$\overline{F}(y_1,..., y_n) = \prod_{i=1}^{n} \overline{F}(y_i)$$

is Schur-concave.

Consider a series system composed of n independent Weibull components with common shape parameter α. Then it is easy to see, in the light of the above theorem, that

(i) the system age positively if α >1,

(ii) the system age negatively if α <1, and,

(iii) the system has no age if α =1.

Hence α is the aging factor for the system.

We shall now prove that the IFRA-ness of a component/system is measured by α. To this end, we give the following definition.

Definition 2.2 (see[6]): Let F and G be continuous life distributions. We say, F is more IFRA than G if $M_F(b) > M_G(b)$ for all 0<b<1, where $M_F(b)$ is defined by

$$M_F(b) = \int_0^{\infty} \overline{F}(bx)dF(x) \tag{2.1}$$

Theorem 2.2: Consider two series systems $\sum_1 and \sum_2$. Let $\sum_i$ be composed of n independent Weibull components with common shape parameter $\alpha, i=1,2.$ Then $\sum_1$ is more IFRA than $\sum_2$ if and only if $\alpha_1 > \alpha_2$.

Proof: Let $X_1,..., X_n$ be the independent lifetimes of n components of the series system. Let X_i follow Weibull $W(\alpha, \beta_j)$, j=1,…,n. Now if $Y = \min(X_1,..., X_n),$ then

$$\overline{F}(x) = P(Y > x) = \exp\left\{-\left(\sum_{i=1}^{n} \frac{1}{\beta_i^{\alpha}}\right) x^{\alpha}\right\}. \tag{2.2}$$

Now, form (2.1), we have

$$M_F(b) = \frac{1}{b^{\alpha} + 1}, \tag{2.3}$$

which is an increasing function of α, for 0<b<1. Hence the theorem follows.

3. The likelihood Function

Consider n series systems each having J components. The random life-lengths of the j-th component in the i-th system, i=1,…,n and j=1,…,J, is denoted by T_{ij}. Let T_i denote the life-length of the i-th system so that

$$T_i = \min(T_{i1},..., T_{iJ}) \tag{3.1}$$

for i=1,…,n. We assume that T_{ij}'s are independent. The random variables $T_{1j},..., T_{nj}$ constitute a random sample from a common absolutely continuous distribution $F_j(t;\theta_j)$ with probability density $f_j(t;\theta_j)$. Let $\overline{F}_j(t;\theta_j) = 1 - F_j(t;\theta_j)$ denote the reliability (survival) function of the j-th component at time t.

After the failure of the i-th system at time t_i, an analysis of the possible cause of failure may be carried out and as a result, a non-empty set, say s_i, is identified which is known to contain the true cause of failure. Before the analysis, s_i is unknown and considered random. Such a random set is denoted by S_i. Let K_i be the random index of the components causing the failure of the i-th system. (K_i is assumed to be unique). Only when $s_i = \{j\}$, we will know that $K_i = j$, in any other case, we can only state that K_i ∈ s_i (i.e., the true cause of failure is masked from our knowledge).

The observations $(T_i, S_i), i = 1,..., n$ are assumed to be independent and the observed data is expressed as $(t_i, s_i), i = 1,..., n$.

The maximum likelihood estimate can now be obtained by maximizing the following expression (see[3]):

$$\prod_{i=1}^{n}\left\{\sum_{j\in S_j}\left(f_j(t_i)\prod_{\substack{s=1\\ s\neq j}}^{J}\overline{F}_s(t_i)\right)\right\} \tag{3.2}$$

4. Consistency and Asymptotic Normality

The likelihood (3.2) seems to be intractable analytically when $J \geq 3$. Therefore, we consider a series system with two independent $W(\alpha, \beta_j)$, j=1,2, components.

Suppose n_1, n_2, n_{12} are the number of observations $(t_i, s_i), i = 1,\ldots,n,$ such that $s_i = \{1\}, s_i = \{2\}, and\ s_i = \{1,2\}$ respectively. From (3.2) one can obtain the log-likelihood function

$$\sum_{i=1}^{n} \ln\left(\alpha t_i^{\alpha-1}\right) - n1 \ln(\beta_1^{\alpha}) - n2 \ln(\beta_2^{\alpha}) + n12 \ln\left(\frac{1}{\beta_1^{\alpha}} + \frac{1}{\beta_2^{\alpha}}\right) - \left(\frac{1}{\beta_1^{\alpha}} + \frac{1}{\beta_2^{\alpha}}\right)\sum_{i=1}^{n} t_i^{\alpha} \quad (4.1)$$

The MLEs are easily shown to be

$$\hat{\beta}_1 = \left[\frac{\sum_{i=1}^{n} t_i^{\alpha}}{n1 + n12 \frac{n1}{n1+n2}}\right]^{1/\alpha} \quad (4.2)$$

$$\hat{\beta}_2 = \left[\frac{\sum_{i=1}^{n} t_i^{\alpha}}{n2 + n12 \frac{n2}{n1+n2}}\right]^{1/\alpha}$$

It is known that

$$\frac{n_j}{n} \xrightarrow{P} P(S_i = \{j\}), j = 1,2; i = 1,\ldots,n \quad (4.3)$$

(see[5], p.356).

We have for j=1,2,

$$P(S_i = \{j\}) = P(S_i = \{j\} \cap (K_i = j))$$
$$= P(S_i = \{j\} / (K_i = j)) P(K_i = j)$$
$$= P(S_i = \{j\} / (K_i = j)) \frac{\beta_j^{-\alpha}}{\beta_1^{-\alpha} + \beta_2^{-\alpha}}. \quad (4.4)$$

Thus,

$$\frac{n_j}{n} \xrightarrow{P} P(S_i = \{j\} / (K_i = j)) \frac{\beta_j^{-\alpha}}{\beta_1^{-\alpha} + \beta_2^{-\alpha}}, j = 1,2. \quad (4.5)$$

By WLLN

$$\frac{\sum_{i=1}^{n} t_i^{\alpha}}{n} \xrightarrow{P} \frac{1}{\beta_1^{-\alpha} + \beta_2^{-\alpha}} \quad (4.6)$$

Therefore,

$$\frac{n}{n_1 + n_{12}\left(\frac{n_1}{n_1 + n_2}\right)} = 1 + \frac{\frac{n_2}{n}}{\frac{n_1}{n}} \quad (\text{since } n_1 + n_2 + n_{12} = n)$$

$$\xrightarrow{P} \frac{\beta_1^{-\alpha} + \beta_2^{-\alpha}}{\beta_1^{-\alpha}}, \quad (4.7)$$

which follows under the assumption (see[3]) that

$$P(S_1 = \{1\} / K_1 = 1) = P(S_2 = \{2\} / K_2 = 2) \quad (4.8)$$

Hence, by the Slutskey's Lemma,

$$\hat{\beta}_1 \xrightarrow{P} \beta_1$$

Similarly, one can show

$$\hat{\beta}_2 \xrightarrow{P} \beta_2 .$$

Thus, $\hat{\beta}_j$ is a consistent estimate of $\beta_j, j = 1,2.$

Now we shall establish the asymptotic normality of $\beta_j^{\alpha}, j = 1,2.$ By the Lindeberg-Levy Central Limit Theorem, we have

$$\frac{\sum_{i=1}^{n} t_i^{\alpha}}{n} \quad is \quad AN\left(\frac{1}{\beta_1^{-\alpha} + \beta_2^{-\alpha}}, \frac{1}{n}\frac{1}{\left(\beta_1^{-\alpha} + \beta_2^{-\alpha}\right)^2}\right)$$

Hence, from (4.7) and the Slutskey's lemma, we have

$$\hat{\beta}_j^{\alpha} \quad is \quad AN\left(\beta_j^{\alpha}, \frac{\beta_j^{2\alpha}}{n}\right), j = 1,2.$$

5. Conclusions

So far the masked data have been studied for the system composed of exponential components. Though there are some papers (see[8], for example) addressing Weibull distribution, no attention is paid to the aging aspect of the situation. In this paper, we have considered a simple series system composed of two Weibull components having the common shape parameter. We have taken the shape parameters to be the same, because (as shown in this paper) it is the aging factor of interest and it characterizes the IFRA-ness of a system. So, by taking the shape parameters to be common, we are basically assuming that the system is consisting of components with the same amount of IFRA-ness.

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