(1,2) - Domination in Line Graphs of $C_n$, $P_n$ and $K_{1,n}$

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Abstract In this paper we discuss the (1,2) - domination in line graphs of $C_n$, $P_n$ and $K_{1,n}$,(1,2)-domination number of paths, cycles, star graphs are found and compared it with the usual domination number. Also we find the domination number of line graphs of these graphs.

Keywords Dominating Set, Domination Number, (1,2) - dominating Set, (1,2) - domination Number

1. Introduction

Let $G = (V,E)$ be a simple graph. A subset $D$ of $V$ is a dominating set of $G$ if every vertex of $V - D$ is adjacent to a vertex of $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.[1,2]

A (1,2) - dominating set in a graph $G = (V,E)$ is a set $S$ having the property that for every vertex $v$ in $V - S$ there is at least one vertex in $S$ at distance 1 from $v$ and a second vertex in $S$ at distance almost 2 from $v$. The order of the smallest (1,2) - dominating set of $G$ is called the (1,2) - domination number of $G$ and we denote it by $\gamma_{(1,2)}$.[7]

The line graph $L(G)$ of a graph $G = (V,E)$ is a graph with vertex set $E(G)$ in which two vertices are adjacent if and only if the corresponding edges in $G$ are adjacent.[8]

2. (1,2) - domination in Paths

Throughout this paper $P_n$ denotes a path on $n$ vertices, $C_n$ denotes a cycle with $n$ vertices and $K_{1,n}$ is a star graph.

Consider the following paths

1. $P_3$

2. $P_4$

3. $P_5$

4. $P_6$

5. $P_7$
\{2,3,4,5,6\} is a (1,2) - dominating set. \\
\(\gamma(1,2) = 5\)

\[\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}\]

\{2,3,4,5,6,7\} is a (1,2) - dominating set. \\
\(\gamma(1,2) = 6\)

\[\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}\]

\{2,3,4,5,6,7,8\} is a (1,2) - dominating set. \\
\(\gamma(1,2) = 7\)

\[\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}\]

\{2,3,4,5,6,7,8,9\} is a (1,2) - dominating set. \\
\(\gamma(1,2) = 8\)

From the above examples we have the following theorem.

**Theorem 2.1**

(1,2) - domination number of a path graph \(P_n\), for \(n \geq 4\) is \(n-2\).

**Proof:**

Let \(P_n\) be a path with \(n\) vertices \(v_1, v_2, \ldots, v_n\). Then \(v_2, v_3, \ldots, v_{n-1}\) are of degree 2, \(v_1\) and \(v_2\) are of degree 1. That is \(n-2\) vertices are of degree 2. Each vertex \(v_i\) is adjacent to \(v_{i+1}\). Therefore, \(v_i\)'s are at distance one from \(v_{i+1}\). Each vertex \(v_{i+2}\) is at distance 2 from \(v_i\). So to form a (1,2) - dominating set we have to include all those vertices of degree 2. But there are \(n-2\) vertices of degree 2.

Hence \(\gamma(1,2) = n-2\).

The following lemma is due to Gray Chartrand and Ping Zhang.[3]

**Lemma 1.** \(\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil\)

The following theorem gives the relationship between domination number and (1,2)-domination number.

**Theorem 2.2**

The domination number of the path \(P_n\) is less than (1,2) domination number.

**Proof:**

We have \(\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil\)

In a graph \(G\), domination number is less than or equals (1,2) domination number[6]. Let \(G\) be a graph and \(D\) be its dominating set. Then every vertex in \(V-D\) is adjacent to a vertex in \(D\). That is, in \(D\) for every vertex \(u\), there is a vertex which is at distance 1 from \(u\). But it is not necessary that there is a second vertex at distance atmost 2 from \(u\). So if we find a (1,2)-dominating set, it will contain more vertices or atleast equal number of vertices than the dominating set. So the domination number is less than or equal to (1,2)-domination number. In particular, for paths domination number is less than(1,2) domination number. Hence the theorem.

3. (1,2)-domination in Cycles

Consider the following cycles

\[\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}\]

\(\{2,3\}\) is a (1,2) - dominating set. \\
\(\gamma(1,2) = 2\)

\[\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}\]

\(\{1,3,4\}\) is a (1,2) - dominating set. \\
\(\gamma(1,2) = 3\)
\[ \gamma_{(1,2)}(C_n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ n - 2, & \text{if } n \text{ is odd} \end{cases} \]

Case 2: \( n \) is odd

If \( n \) is odd, we remove one vertex \( v_1 \), then the other \( n-1 \) vertices form a path \( P_{n-1} \) and \( n-1 \) is even. But \( (1,2) \)-domination number of \( P_{n-1} \) is \( n-3 \). These \( n-3 \) vertices and the vertex \( v_1 \) from a \((1,2)\) dominating set. Hence the cardinality of the \((1,2)\) dominating set is \( n-3+1 \), that is \( n-2 \). Hence \( \gamma_{(1,2)} = n-2 \), if \( n \) is odd.

**4. (1,2)-domination in Star Graphs**

Consider the following star graphs.

\[ \{1,2\} \text{ form a } (1,2) \text{- dominating set. } \gamma_{(1,2)} = 2 \]

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\[ \gamma_{(1,2)} = 2 \]

**Theorem 4.1**
For any star \( K_{1,n} \), \( \gamma_{(1,2)}(K_{1,n}) = 2 \).

**Proof:**
In a star \( K_{1,n} \), there are \( n+1 \) vertices \( v, v_1, v_2, \ldots, v_n \). \( v \) is adjacent to all other vertices \( v_1, v_2, \ldots, v_n \). \{\( v_1, v_2, \ldots, v_n \)\} form an independent set. Each of \( v_1, v_2, \ldots, v_n \) are at a distance 1 from \( v \) and each of \( v_2, v_3, \ldots, v_n \) are at a distance 2 from \( v_1 \). So we can form a \((1,2)\) dominating set as \{\( v, v_1 \)\}. Clearly the cardinality is two. Hence \( \gamma_{(1,2)}(K_{1,n}) = 2 \).

**5. (1,2) - domination in the Line Graph of \( P_n, C_n, K_{1,n} \).**

In this section first we discuss the line graphs of paths, cycles and star graphs.

Consider the paths and the corresponding line graphs

\[ P_3 : \begin{array}{c}
1 \\
2 \\
3 \\
P_3
\end{array} \quad \begin{array}{c}
1 \\
2 \\
L(P_3)
\end{array} \]

\[ P_4 : \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
P_4
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3 \\
P_3
\end{array} \]

\[ P_5 : \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
P_5
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
P_4
\end{array} \]

Next consider the cycles and the corresponding line graphs

\[ C_3 : \begin{array}{c}
1 \\
2 \\
3 \\
L(C_3)
\end{array} \]

\[ C_4 : \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
L(C_4)
\end{array} \]

Consider the following star graphs and the corresponding line graphs
From the above examples and from [7] we have the following observations

The line graph of $P_n$, $L(P_n)$ has

(i) $n-1$ vertices and $n-2$ edges

(ii) 2 vertices of degree 1 and $n-3$ vertices of degree 2.

Hence $L(P_n) = P_{n-1}$.

The line graph of $C_n$ has

(i) $n$ vertices and $n$ edges

(ii) Each vertex is of degree 2

(iii) The graph is cyclic

Hence $L(C_n) = C_n$.

The line graph of $K_{1,n}$ has

(i) $n-1$ edges and $n$ vertices

(ii) Each of the $n$ vertices is of degree 2.

(iii) The graph is cyclic

$L(K_{1,n}) = C_n$ for $n \geq 3$.

**Theorem 5.1**

$(1,2)$ - domination number of $L(P_n)$ is $n-3$.

Proof:

$P_n$ has $n$ vertices and $n-1$ edges and $L(P_n)$ is $P_{n-1}$ with $n-1$ vertices and $n-2$ edges. Then by theorem 2.1, $(1,2)$ - domination number of $P_{n-1}$ with $n-3$. Hence $(1,2)$ - domination in the line graph of $L(P_n)$ is $n-3$.

**Theorem 5.2**

$(1,2)$ - domination number of $L(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ n-2 & \text{if } n \text{ is odd} \end{cases}$
Proof:
The line graph of $C_n$, $L(C_n)$ is $C_n$ itself. So we can apply theorem 3.1

Hence $\gamma_{(1,2)}(L(C_n)) = \begin{cases} 
\frac{n}{2}, \text{if } n \text{ is even} \\
n-2, \text{if } n \text{ is odd}
\end{cases}$

**Theorem 5.3**

$(1,2)$ - domination number of $L(K_{1,n})$ is same as that of $C_n$.  

Proof:
The line graph of $K_{1,n}$ is $C_n$. Then by theorem 3.2,

$\gamma_{(1,2)}(L(K_{1,n})) = \begin{cases} 
\frac{n}{2}, \text{if } n \text{ is even} \\
n-2, \text{if } n \text{ is odd}
\end{cases}$

6. Conclusions

In this paper the $(1,2)$-dominating sets and $(1,2)$ - domination numbers of some standard graphs $P_n$, $C_n$ and $K_{1,n}$ have been discussed and the results extended to the line graphs of those graphs. It is also found that $(1,2)$ - domination number of $C_n$ and $K_{1,n}$ are same.

**REFERENCES**


[7] Steve Hedetniemi, Sandee Hedetniemi, $(1,2)$ - Domination in Graphs.