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Solution of Bratu's Equation by He's Variational Iteration Method

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Abstract The purpose of this paper is to use He's variational iteration methodfor solving Bratu's boundary value problem, using only three terms in series expansion of nonlinear part. The method converges rapidly and approximates the exact solution very accurately. Two special cases of the problem are illustrated by using two iterates of the recursive scheme and the numerical results and conclusions will be presented.

Keywords Variational Iteration Method, Nonlinear Equations, Bratu's Problem

1. Introduction

One of the problems that had been attacked by several authors is Bratu's boundary value problem which is given by

$$y'' = -\lambda e^{y} \quad (1)$$

$$y(0) = 0 \text{ and } y(1) = 0, \quad (2)$$

where $\lambda > 0$.A typical example occurs in the theory of the electric charge around a hot wire and also in certain problems of solid mechanics. The Bratu's problem in one-dimensional planner coordinates has two known, bifurcated solutions for values of $\lambda < \lambda_c$, no solution for $\lambda > \lambda_c$ and a unique solution when $\lambda = \lambda_c$.The value of λ_c is related to the fixed point of hyperbolic contangent function and satisfies the equation $1 = 0.25\sqrt{2\lambda}\sinh(0.25\theta)$.

The exact solution of (1) and (2) is given by

$$y(x) = -2\log\left\{\frac{\cosh[0.5(x-0.5)\theta]}{\cosh(0.25\theta)}\right\},$$

provided that θ is the solution of $\theta = \sqrt{2\lambda}\cosh(0.25\theta)$.With given boundary conditions it is not possible to solve (1) by elementary methods.Various kinds of analytical methods and numerical methods were used to solve this equation[1-6]. For example, Bellman and Kalaba find substantial agreement of $y_2(x)$ to exact solution by applying quasilinearization method [5] and S. A. Khuri used a Laplace transform numerical technique for solving this problem [3].

It been shown that the variational iteration method is a very efficient tool for solving various kinds of nonlinear ordinary and partial differential equations[7-12]. Itbeen used to solve the Fokker Planck equation, the Lane-Emden differential equation, the Klein-Gordon partial differential equations, the Cauchy reaction-diffusion problem, the biological population model. For more applications of the method the interested reader is referred to [7]. It also is useful to solve integral equations [13-15].

To illustrate the basic idea of the method, we consider the differential equation

$$\mathcal{L}y + Ny = f(x) \quad (3)$$

where $\mathcal{L}$ and N are linear and nonlinear operators respectively, and $f(x)$ is the source inhomogeneous term.

The variational iteration method presents a correction functional for Eq. (3)
in the form

$$y_{n+1}(x) = y_n(x) + \int_0^x \bar{\lambda}[\mathcal{L}y_n(\xi) + N\tilde{y}_n(\xi) - f(\xi)]d\xi,$$

where $\bar{\lambda}$ is a general Lagrange multiplier[3], which can be optimally found via variational theory and $\tilde{y}_n$ is a restricted variation which means, $\delta\tilde{y}_n = 0$.

2. Procedure

RecallBratu's boundary value problem

$$y'' = -\lambda e^{y} \quad (1)$$

$$y(0) = 0 \text{ and } y(1) = 0, \quad (2)$$

We use Taylor series expansion for nonlinear part and write

$$y'' + \lambda e^{y} = y'' + \lambda\sum_{k=0}^{\infty}\frac{y^k}{k!} \cong y'' + \lambda\left(1 + y + \frac{y^2}{2}\right).$$

Thus

$$y_{n+1}(x) = y_n(x) + \int_0^x \bar{\lambda}\left[y_n'' + \lambda\left(1 + y_n + \frac{y_n^{\,2}}{2}\right)\right]d\xi.$$

Impose $y_0(x) = kx$ (one may use $y_0(x) = k(x-1)$ and apply $y_n(0) = 0$ to find k), we obtain

$$y_1(x) = kx + \lambda\int_0^x (\xi - x)\left(1 + k\xi + \frac{k^2\xi^2}{2}\right)d\xi$$

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Integrating by parts leads to

$$y_1(x) = kx - \frac{\lambda x^2}{2!} - \frac{\lambda k x^3}{3!} - \frac{\lambda k^2 x^4}{4!}.$$

We have

$$y_2(x) = y_1(x) + \int_0^x (\xi - x)\left[y_1'' + \lambda\left(1 + y_1 + \frac{y_1^2}{2}\right)\right] d\xi.$$

Therefore,

$$y_2(x) = kx - \frac{\lambda x^2}{2!} - \frac{\lambda k x^3}{3!} - \frac{\lambda k^2 x^4}{4!} + \lambda\int_0^x (\xi - x)\begin{bmatrix} -\frac{\lambda\xi^2}{2} - \frac{2\lambda k\xi^3}{3} + \frac{\lambda}{24}(3\lambda - 5k^2)\xi^4 \\ +\frac{\lambda k}{24}(2\lambda - k^2)\xi^5 + \frac{5\lambda^2 k^2 \xi^6}{144} \\ +\frac{\lambda^2 k^3 \xi^7}{144} + \frac{\lambda^2 k^4 \xi^8}{1152} \end{bmatrix} d\xi.$$

Again, we use integration by parts. We obtain

$$y_2(x) = kx - \frac{\lambda x^2}{2!} - \frac{\lambda k x^3}{3!} - \frac{\lambda(k^2 - \lambda)x^4}{4!} + \frac{4\lambda^2 k x^5}{5!} + \frac{\lambda^2(5k^2 - 3\lambda)x^6}{6!} + \frac{5\lambda^2 k(k^2 - 2\lambda)x^7}{7!} - \frac{25\lambda^3 k^2 x^8}{8!} - \frac{35\lambda^3 k^3 x^9}{9!} - \frac{35\lambda^3 k^4 x^{10}}{10!}. \quad (4)$$

Since $y(1) = 0$, with a given value for λ, we can find four values for k. With appropriate choose of k, $y(x)$ can be approximated for $x \in (0, 1)$.

3. Examples

In this section we examine the above procedure for $\lambda = 1$ and $\lambda = 2$. Applying $\lambda = 1$ to Eq. (4) and simplify it, we come to

$$35k^4 - 3250k^3 + 128250k^2 - 3137760k + 1678320 = 0.$$

Solving this equation by a numerical method, gives

$$k_1 = 0.546936690480377 \quad ,$$
$$k_2 = 55.687874088793876 .$$

Since $y(1) = 0$, with two decimal points we may choose, $k \cong 0.55$ then

$$y_2(x) = 0.55x - \frac{x^2}{2!} - \frac{0.55x^3}{3!} + \frac{0.6975x^4}{4!} + \frac{2.2x^5}{5!} - \frac{1.4875x^6}{6!} - \frac{4.668125x^7}{7!} - \frac{7.5625x^8}{8!} - \frac{5.823125x^9}{9!} - \frac{3.202717x^{10}}{10!}.$$

We tested it for values of $x = 0.1$, $x = 0.2, \ldots, x = 0.9$, respectively and the results are given in Table 1.

For $\lambda = 2$ to Eq. (4), we come to

$$y_2(x) = kx - \frac{2x^2}{2!} - \frac{2kx^3}{3!} - \frac{2(k^2 - 2)x^4}{4!} + \frac{16kx^5}{5!} + \frac{4(5k^2 - 6)x^6}{6!} + \frac{20k(k^2 - 4)x^7}{7!} - \frac{200k^2x^8}{8!} - \frac{280k^3x^9}{9!} - \frac{280k^4x^{10}}{10!}. \quad (5)$$

If we apply $y(1) = 0$ to Eq. (5), we obtain

$$28k^4 - 1160280k^3 + 21960k^2 - 284544k + 314496 = 0.$$

Roots of this equation are

$$k_1 = 1.211500000137994 \quad ,$$
$$k_2 = 25.631365803713052 \quad .$$

Our appropriate choose will be $k \cong 1.2$. We tested it similar to case of $\lambda = 1$. The results are given in Table 2.

Table 1.

Approximate solution	Exact solution	Absolute error
0.049911420756181	0.049846791245413	6.46295×10^{-5}
0.089318888767090	0.089189934628823	1.28954×10^{-4}
0.117803234965962	0.117609095767941	1.94139×10^{-4}
0.135054959713536	0.134790253884190	2.64706×10^{-4}
0.140890707842786	0.140539214400472	3.51493×10^{-4}
0.135266464525427	0.134790253884190	4.76211×10^{-4}
0.118286700479587	0.117609095767941	6.77605×10^{-4}
0.090208593451626	0.089189934628823	1.01866×10^{-3}
0.051440344054266	0.049846791245413	1.59355×10^{-3}

Table 2.

Approximate solution	Exact solution	Absolute error
0.110602751900ß23	0.114410743267745	3.65852×10^{-3}
0.199192299784140	0.206419116487609	7.2268×10^{-3}
0.259968498111216	0.273879311825552	1.39108×10^{-3}
0.297235353011736	0.315089364225670	1.78540×10^{-2}
0.307894933035714	0.328952421341114	2.10574×10^{-2}
0.291925007727470	0.315089364225670	2.31644×10^{-2}
0.250206159286683	0.273879311825552	2.36732×10^{-2}
0.184540055769673	0.206419116487609	2.18791×10^{-2}
0.097561558040441	0.114410743267745	1.68492×10^{-2}

4. Conclusions

Results in this paper show the efficiency of variational iterative method and is quite reliable. It can be observed that our approximations are accurate using only two iterates and avoid the much computation using Adomian polynomials[1, 3] or quasilinearization method [5]. There is no doubt, if more terms of Taylor series expansion, or more iterates were used, we obtain much better results. This approach also can be used for Poisson-Boltzman's equation in one-dimensional planar coordinates if we choose $\lambda = -1$.

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