Modified Variational Iteration Method for the Solution of a Class of Differential Equations

M.O Olayiwola[1,*], F.O Akinpelu[2], A.W Gbolagade[1]

[1]Department of Mathematical & Physical Sciences, College of Science Engineering & Technology, Osun State University, Osogbo, Nigeria
[2]Department of Pure & Applied Mathematics,Faculty of Pure & Applied Sciences, Ladoke Akintola University of Technology, Ogbomoso, Nigeria

Abstract In this paper, a Modified Variational Iteration Method (MVIM) for the solution of a differential equation of Bratu-type is presented. The method converges to the exact solution after an iteration. This shows that the method is efficient for this class of initial and boundary value problems.

Keywords Modified Variational Iteration Method, Lagrange Multiplier, Taylor's Series, Bratu-type Differential Equation

1. Introduction

Differential equations as a subject are a deductive science and a branch of Mathematics which have strong roots in physical problems such as Physics and Engineering. This subject also derives much of its power and beauty from the variety of its applications

This equations play a crucial role in applied mathematics and physics. The results of solving such equations can guide authors to know the described process deeply. But it is difficult to obtain the exact solution for some of these problems. In recent decades, there has been great development in the numerical methods for the solution of ordinary and partial linear and nonlinear differential equations.

Bratu-type differential equation is an initial and boundary value problem in one-dimensional planar coordinate. It is used to model a combustion problem in a slab such as fuel ignition of the thermal combustion theory and in the Chandrasekhar model of the expansion of the universe[11]. It also used to simulate a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction.

The Bratu-type is of the form:

$$\frac{d^2u}{dx^2} + \lambda\, e^u = 0, \, 0 < x < 1 \tag{1}$$

The exact solution is

$$u(x) = 2\ln\left[\frac{\cosh\left((x-\frac{1}{2})\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right] \tag{2}$$

Where $\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$ \hfill (3)

The equation has zero, one or two solution when $\lambda > \lambda_c$, $\lambda = \lambda_c$, $\lambda < \lambda_c$ respectively. The λ_c is the critical value which satisfies

$$I = \frac{1}{4}\sqrt{2\lambda_c}\sinh\left(\frac{\theta}{4}\right) \tag{4}$$

It has been shown that $\lambda_c = 3.51380719$ see[1,11,12]

Several authors have presented various numerical approach to the solution of Bratu-type differential equations[1,2,6,11,12].

The variational iteration method was proposed by J.H He[4-5]. In this paper a Modivied Variational Iteration Method proposed by Olayiwola M O[7-9] is presented for the solution of Bratu-type differential equation. MVIM is the combination of VIM and the Taylor's polynomial.

2. Variational Iteration Method

To illustrate the basic concept of the VIM, we consider the following general nonlinear partial differential equation.

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \tag{5}$$

where L is a linear time derivative operator, R is a linear operator which has partial derivative with respect to x, N is a nonlinear operator and g is an inhomogeneous term.

According to VIM, we can construct a correct fractional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda [Lu_n + R\tilde{u}_n + N\tilde{u}_n - g] d\tau \quad (6)$$

where λ is a Lagrange multiplier which can be identified optimally via variational iteration method. The subscript n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation i.e, $\delta\tilde{u}_n = 0$. The successive approximation $u_{n+1}, n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by:

$$u = \lim_{n \to \infty} u_n \quad (7)$$

In Modified Variational Iteration Method, equation (6.0) becomes:

$$u_0(x,t) = g_0(x) + tg_1(x) + t^2 g_2(x) \quad (8)$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda [Lu_n + R\tilde{u}_n + N\tilde{u}_n - g] d\tau \quad (9)$$

where $g_2(x)$ can be found by substituting for $u_0(x,t)$ in (5.0) when $t = 0$.

3. Stationary Conditions

The simplest problem of the calculus of variation is to determine a function:

$$y = f(x) \quad (10)$$

for which the value of a given functional

$$f = \int_{x_1}^{x_2} F(y,y':x)dx + g_1(x)y\big|_{x=x_1} - g_2(x)y\big|_{x=x_2} = 0 \quad (11)$$

is a minimum or maximum.

The extremum condition (stationary condition) of the functional (11) requires that

$$\delta f = \delta \int_{x_1}^{x_2} F(y,y':x)dx + g_1\delta y\big|_{x=x1} - g_2\delta y\big|_{x=x_2} \quad (12)$$

For arbitrary δy, we have

$$\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0 \quad (13)$$

and the boundary conditions

$$\frac{df}{dy'}(x_1) - g_1(x_1) = 0 \quad and \quad \frac{df}{dy'}(x_2) - g_2(x_2) = 0 \quad (14)$$

4. Derivation of λ

Consider (9) of the form:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left[\frac{\partial u}{\partial t}\right] d\tau \quad (15)$$

Making (15) stationary

$$u_{n+1} = u_n + u\lambda - \tilde{u}\lambda' + \int \tilde{u}\lambda'' d\tau \quad (16)$$

$$\frac{\partial u_{n+1}}{\partial u_n} = 1 + \lambda\big|_{\tau=t} = 0 \quad (17)$$

$$\frac{\partial u_{n+1}}{\partial \tilde{u}_n} = -\lambda\big|_{\tau=t} \quad (18)$$

Solving (17-18), we have

$$\lambda = -1 \quad (19)$$

Consider equation (9) of the form:

$$mu''(x,t) + Ru(x,t) + Nu(x,t) = g(x,t). \quad (20)$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left[mu''(x,\tau) + R\tilde{u}_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,\tau)\right] d\tau \quad (21)$$

Making (21) stationary, we have:

$$\delta u_{n+1} = \delta u_n + \delta\left(\lambda m u'_n - mu_n\lambda' + \int mu_n\lambda'' d\tau\right) \quad (22)$$

This yields the following stationary condition

$$1 - m\lambda'\big|_{\tau=t} = 0 \quad (23)$$

$$\lambda m\big|_{\tau=t} = 0 \quad (24)$$

$$m\lambda'' = 0 \quad (25)$$

Solving (23-25), we have

$$\lambda = \frac{1}{m}(\tau - t) \quad (26)$$

Equation (21) becomes

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \frac{1}{m}(\tau - t)\left[mu''_n(x,\tau) + R\tilde{u}_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,\tau)\right] d\tau \quad (27)$$

5. Convergence of the MVIM

Let

$$\frac{\partial^{n1}}{\partial t^{n1}}U(x,s) + \frac{\partial^{m1}}{\partial x^{m1}}U(x,s) + AU(x,s) = F(s,t) \quad (28)$$

Theorem 1: Let $U(x,t), (x,t) \in R = [0,L]X[0,T]$ be the exact solution of (9) and $U_m(x,t)$ be the solution of the sequence

$$U_{m+1}(x,t) = U_m(x,t) - \int_0^t e^{-(s-t)D}\left[\frac{\partial}{\partial s}U_m(x,s) + \frac{\partial}{\partial x}U_m(x,s) - AU_m(x,s) - F(s,x)\right] ds$$

With $U_0(x,t) = U_0$

If

$$E_m(x,t), = U_m(x,t) - U(x,t) \ and \ \left\| \frac{\partial}{\partial x} E_m(x,t) \right\|_2 \leq \left\| E_m(x,t) \right\|_2$$

then the functional sequence $\{U_m(x,t)\}_{m=1}^{m=\infty}$ converges

to $U(x,t)$ [10].

Theorem 2: Taylor's Theorem:
A polynomial is a function of the form:

$$P(x) = a_0 + a_1(x - x_0) + ... + a_n(x - x_0)^n \qquad \text{Where}$$

$a_0, ..., a_n$ and x_0 are constants. The polynomial is said to be written in powers of $x - x_0$, and is of degree n if $a_n \neq 0$

Lemma 1: If f is differentiable at x_0, then

$$f(x) = f(x_0) + [f'(x_0) + E(x)](x - x_0)$$

where E is defined on a neighborhood of x_0 and $\lim_{x \to x_0} E(x) = E(x_0) = 0$

Theorem 3: If $f^n(x_0)$ exists for some integer $n \geq 1$ and T_n is the nth Taylor polynomial of f about x_0, then the limit exists.

Since Taylor's series converges and VIM converges, then MVIM converges.

6. Application of MVIM

We present a more stable and reliable method for the solution of the form:

$$\frac{d^2 u}{dx^2} + \lambda \, e^u = 0, 0 < x < 1 \qquad (29)$$

$$u(0) = u^1(0) = u(1) = 0$$

Where λ can take any value.
Applying (8.0 - 9.0) in (10.0) we have

$$u_0(x) = \frac{\lambda}{2} x^2 \qquad (30)$$

Using Maple to implement MVIM, we obtained

$$u_1(x) = -\frac{\lambda}{2} x^2 + \frac{\lambda^2}{24} x^4 - \frac{\lambda^3}{180} x^6 + \frac{17\lambda^4}{20160} x^8 -$$
$$\frac{7\lambda^5}{64800} x^{10} + \frac{79\lambda^6}{5443200} x^{12} - \frac{5041\lambda^7}{2490808320} x^{14} + ... \qquad (31)$$

Test Case 1: Vahidi and Hasanzade[2] considered $\lambda = -2$. The problem becomes:

$$\frac{d^2 u}{dx^2} - 2 \, e^u = 0, 0 < x < 1 \qquad (32)$$

Vahidi and Hasanzade[2]. (11.0) becomes:

$$u_1(x) = x^2 + \frac{1}{6} x^4 + \frac{2}{45} x^6 + \frac{17\lambda^4}{1260} x^8 +$$
$$\frac{7}{2025} x^{10} + \frac{79}{85050} x^{12} + \frac{5041}{19459440} x^{14} + ... \ .. \qquad (33)$$

Table 1. Error comparison of the methods

X	U_MVIM	U_EXACT	U_MVIM_ERROR	U_VIM	U_VIM_ERROR	U_RADM
0	0.0000000000E+00	0.0000000000E+00	0.0000000000E+00	0.0000000000E+00	0.0000000000E+00	0.0000000000E+00
0.1	1.0016711246E-02	1.0016711246E-02	9.2151980491E-14	1.0016711246E-02	2.1858730104E-13	9.3045000000E-14
0.2	4.0269546009E-02	4.0269546105E-02	9.6200672428E-11	4.0269545881E-02	2.2402885796E-10	9.7158000000E-11
0.3	9.1383306132E-02	9.1383311852E-02	5.7198028536E-09	9.1383298913E-02	1.2939068289E-08	5.7790000000E-09
0.4	1.6445793202E-01	1.6445803815E-01	1.0613039200E-07	1.6445780737E-01	2.3077865857E-07	1.0727000000E-07
0.5	2.6116743284E-01	2.6116848089E-01	1.0480466828E-06	2.6116631082E-01	2.1700651015E-06	1.0595000000E-06
0.6	3.8392334169E-01	3.8393033884E-01	6.9971523368E-06	3.8391665406E-01	1.3684782547E-05	7.0675000000E-06
0.7	5.3613556113E-01	5.3617151514E-01	3.5954008103E-05	5.3610552405E-01	6.5991090026E-05	3.6212000000E-05
0.8	7.2262749811E-01	7.2278149362E-01	1.5399550904E-04	7.2251743841E-01	2.6405521358E-04	1.5413000000E-04
0.9	9.5030440084E-01	9.5088488717E-01	5.8048633037E-04	9.4995723943E-01	9.2764774446E-04	5.7446000000E-04

The table above shows the comparison of result obtained by Vahidi[2] u_RADM , Noor and Mohyud-Din[6] u_VIM and u_MVIM .with the exact solution u_EXACT

*Test Case 2:*Noor and Mohyud-Din[6], considerd:

$$\frac{d^2u}{dx^2} - \pi^2\,e^u = 0,\ 0 < x < 1 \qquad (34)$$

where $\lambda = -\pi^2$

This gives:

$$u_1(x) = \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4 + \frac{\pi^6}{180}x^6 + \frac{17\pi^8}{20160}x^8 +$$

$$\frac{7\pi^{10}}{64800}x^{10} + \frac{79\lambda^6}{5443200}x^{12} + \frac{5041\lambda^7}{2490808320}x^{14} + \ldots \ldots \qquad (35)$$

*Test Case 3:*Noor and Mohyud-Din[6], considerd:

$$\frac{d^2u}{dx^2} + \pi^2\,e^u = 0,\ 0 < x < 1 \qquad (36)$$

where $\lambda = +\pi^2$

This gives:

$$u_1(x) = -\frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4 - \frac{\pi^6}{180}x^6 + \frac{17\pi^8}{20160}x^8 -$$

$$\frac{7\pi^{10}}{64800}x^{10} + \frac{79\lambda^6}{5443200}x^{12} - \frac{5041\lambda^7}{2490808320}x^{14} + \ldots \ldots \qquad (37)$$

The exact solutions for (13.0, 15.0, 17.0) are:

$$u(x) = -2\ln(\cos(x)) \qquad (38)$$

$$u(x) = \ln(1 + \sin(1 + \pi x)) \qquad (39)$$

$$u(x) = -\ln(1 + \cos((.5 + x)\pi)) \qquad (40)$$

7. Conclusions

In this study, we have shown that the modified variational iteration method can be successfully applied for finding the solution of a class of differential eqution.

The above table is the comparison of the error results of MVIM result after two iteration, the Noor[6] after six iteration and Vahid[2] after five iteration.

The method does not involved the introduction of any set of algebraic equations that will be solved for another set of variables. The other two cases also converges to the exact solution.

The result shows that the MVIM is a novel approach to the solution of Bratu-type differential equations.

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