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Bernoulli Matrix Approach for Solving Two Dimensional Linear Hyperbolic Partial Differential Equations with Constant Coefficients

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Abstract The purpose of this study is to give a Bernoulli polynomial approximation for thesolution of hyperbolic partial differential equations with three variables and constant coefficients. For this purpose, a Bernoulli matrix approach is introduced. This method is based on taking the truncated Bernoulli expansions of the functions in the partial differential equations. After replacing the approximations of functions in the basic equation, we deal with a linear algebraic equation. Hence, the result matrix equation can be solved and the unknown Bernoulli coefficients can be found approximately. The efficiency of the proposed approach is demonstrated with one example.

Keywords Bernoulli Polynomial Solutions, Two-dimensional Linear Hyperbolic Equation With Constant Coefficients, Bernoulli Matrix Method

1. Introduction

When a mathematical model is formulated for a physical problem, it is often represented by partial differential equations, that are not solvable exactly by analytic techniques. Therefore, onemust resort to approximation and numerical methods. For example,theTelegraph equation (as a hyperbolic partial differential equation) is commonly used in signal analysis for transmission and propagation of electrical signals and also has applications in other fields (see[1] and the references therein). We also note that,the differential equationsaresometimes linear in real world as is shown in papers[2, 3].

The solution of linear hyperbolic partial differential equations clarifies the linear phenomena which occur in many systems like as biology, engineering, aerospace, industry etc. In the recent years, noticeable progress has been made in the construction of the numerical solutions for linear partial differential equations, which has long been a major concern for both mathematicians and physicists.The various methods for solving linear hyperbolic partial differential equations are introducedin[4].

Since the beginning of 1994 several matrix (such as Legendre, Chebyshev, Bessel, Lagurre, Hermite, Bernstein etc.) approachesfor solving linear differential, integral, integro- differential, difference, and integro-difference equations, havebeen presented in many papers by Sezer and Coworkers[4–11].

In this study, and in the light of the above mentioned methods (and by generalizing the Bernoulli matrix method for one dimensional PDEs[12]) we propose a new matrix approach which is based on Bernoulli truncated series in three dimension for solving two dimensional hyperbolic partial differential equations with constant coefficients in the following form

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2}(x,y,t)+2\alpha\frac{\partial u}{\partial t}(x,y,t)+\beta^2 u(x,y,t)\\&+\theta\frac{\partial^2 u}{\partial x^2}(x,y,t)+\eta\frac{\partial^2 u}{\partial y^2}(x,y,t)=f(x,y,t),\\&(x,y,t)\in[0,1]\times[0,1]\times(0,1),\end{aligned}\tag{1}$$

with the initial conditions

$$u(x,y,0)=\psi_1(x,y),\ \frac{\partial u}{\partial t}(x,y,0)=\psi_2(x,y)\tag{2}$$

and the boundary conditions

$$\begin{aligned}&u(x,0,t)=f_0(x,t),\ u(x,1,t)=f_1(x,t),\ t\ge 0,\\&u(0,y,t)=g_0(y,t),\ u(1,y,t)=g_1(y,t),\ t\ge 0,\end{aligned}\tag{3}$$

and the solution is approximated by the following Bernoulli truncatedseries

$$u(x,y,t)\approx u_N(x,y,t)=\sum_{p=0}^{N}\sum_{r=0}^{N}\sum_{s=0}^{N}a_{p,r,s}B_p(x)B_r(y)B_s(t),$$

$$a_{p,r,s}=\frac{1}{p!r!s!}\int_0^1\int_0^1\int_0^1\frac{\partial^{p+r+s}u(x,y,t)}{\partial x^p\partial y^r\partial t^s}dtdydx,\tag{4}$$

so the Bernoulli coefficients to be determined are $a_{p,r,s}$;

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where $f,\ \psi_1,\ \psi_2,\ f_0,\ f_1,\ g_0\ and\ g_1$ are known functions $(\mathrm{p,r,s}=0,\ \ldots,\mathrm{N})$.

The considered partial differential equation (1) arise in connection with various physical and geometrical problems in which the functions involved depend on two or morein-dependent variables, on time t and on one or several space variables[4]. For $\alpha>0,\ \beta>0$ and $\theta=\eta=-1$, equation(1) represents a two-space-dimensional damped wave equation (with a source term). The numerical solution of damped wave equations is of great importance in studying wave phenomena.

2. Fundamental Relations

To obtain the numerical solution of the hyperbolic partial differential equation with thepresented method, we first evaluate the Bernoulli coefficients of the unknown function. For convenience, the solution function (4) can be written in the matrix form

$$u(x,y,t)=\hat{\hat{B}}(x,y,t)A, \tag{5}$$

where $\hat{\hat{B}}(x,y,t)$ is the $1\times(N+1)^3$ matrix defined as

$$\hat{\hat{B}}(x,y,t)=\left[\hat{\hat{B}}_0(x,y,t)\ \hat{\hat{B}}_1(x,y,t)\ \ldots\ \hat{\hat{B}}_N(x,y,t)\right]_{1\times(N+1)^3},$$

$$\hat{\hat{B}}_i(x,y,t)=B_i(t)\hat{B}(x,y)$$

$$\hat{B}(x,y)=\left[\hat{B}_0(x,y)\ \hat{B}_1(x,y)\ \cdots\ \hat{B}_N(x,y)\right]_{1\times(N+1)^2}$$

$$\hat{B}_i(x,y)=B_i(y)B(x);\quad B(x)=\left[B_0(x)\ B_1(x)\ \ldots\ B_N(x)\right]$$

and

$$A=\begin{bmatrix}A_0 & A_1 & \cdots & A_N\end{bmatrix}^T$$

where

$$A_i=\begin{bmatrix}a_{0,0,i} & a_{1,0,i} & \cdots & a_{N,0,i} & a_{0,1,i} & a_{1,1,i} & \cdots \\ a_{N,1,i} & \cdots & a_{0,N,i} & a_{1,N,i} & \cdots & a_{N,N,i} & \end{bmatrix}^T$$

On the other hand, the relations between the matrix $\hat{\hat{B}}(x,y,t)$ and its derivatives are as follows:

$$\begin{aligned}u(x,y,t)^{(n,0,0)}&=\hat{\hat{B}}^{(n,0,0)}(x,y,t)A\\&=\hat{\hat{B}}(x,y,t)\left(\hat{\hat{H}}\right)^n A,\quad n=0,1,2\end{aligned} \tag{6}$$

$$\begin{aligned}u(x,y,t)^{(0,m,0)}&=\hat{\hat{B}}^{(0,m,0)}(x,y,t)A\\&=\hat{\hat{B}}(x,y,t)\left(\hat{M}\right)^m A,\quad m=0,1,2\end{aligned} \tag{7}$$

$$\begin{aligned}u(x,y,t)^{(0,0,s)}&=\hat{\hat{B}}^{(0,0,s)}(x,y,t)A\\&=\hat{\hat{B}}(x,y,t)\left(S\right)^s A,\quad s=0,1,2\end{aligned} \tag{8}$$

where

$$H=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & N\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},\ \hat{H}=\begin{bmatrix}H & 0 & \cdots & 0\\ 0 & H & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & H\end{bmatrix},$$

$$\hat{\hat{H}}=\begin{bmatrix}\hat{H} & 0 & \cdots & 0\\ 0 & \hat{H} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \hat{H}\end{bmatrix}$$

$$M=\begin{bmatrix}0 & I & 0 & \cdots & 0\\ 0 & 0 & 2I & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & NI\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},\hat{M}=\begin{bmatrix}M & 0 & \cdots & 0\\ 0 & M & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & M\end{bmatrix},$$

$$S=\begin{bmatrix}0 & \hat{I} & 0 & \cdots & 0\\ 0 & 0 & 2\hat{I} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & N\hat{I}\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}$$

where I and $\hat{I}$ are $(N+1)\times(N+1)$ and $(N+1)^2\times(N+1)^2$ identity matrices respectively.

By using the relations (6)-(8)we have

$$\begin{aligned}&u^{(m,n,s)}(x,y,t)=\hat{\hat{B}}^{(m,n,s)}(x,y,t)A\\&=\hat{\hat{B}}(x,y,t)\left(\hat{\hat{H}}\right)^m\left(\hat{M}\right)^n\left(S\right)^s A,\quad m,n,s=0,1,2\end{aligned}$$

We can also expand the function $f(x,y,t)$ as a Bernoulli series:

$$\begin{aligned}f(x,y,t)&=\sum_{p=0}^{N}\sum_{r=0}^{N}\sum_{s=0}^{N}f_{p,r,s}B_p(x)B_r(y)B_s(t),\\ f_{p,r,s}&=\frac{1}{p!r!s!}\int_0^1\int_0^1\int_0^1\frac{\partial^{p+r+s}f(x,y,t)}{\partial x^p\partial y^r\partial t^s}dtdydx.\end{aligned} \tag{9}$$

From (9),one can obtain the following matrix form

$$f(x,y,t)=\hat{\hat{B}}(x,y,t)F \tag{10}$$

where

$$F=\begin{bmatrix}F_0 & F_1 & \cdots & F_N\end{bmatrix}^T$$

with

$$F_i=\begin{bmatrix}f_{0,0,i} & f_{1,0,i} & \cdots & f_{N,0,i} & f_{0,1,i} & f_{1,1,i} & \cdots \\ f_{N,1,i} & \cdots & f_{0,N,i} & f_{1,N,i} & \cdots & f_{N,N,i} & \end{bmatrix}^T.$$

Substituting the expressions (5)–(8) and (10) into equation(1) and simplifying the result, we have the matrix equation

$$\left[\left(S\right)^2+2\alpha S+\beta^2\hat{\hat{I}}+\theta\left(\hat{\hat{H}}\right)^2+\eta\left(\hat{M}\right)^2\right]A=F. \tag{11}$$

Briefly, we can write equation(11) in the form

$$\mathrm{WA=F}, \tag{12}$$

where $\hat{\hat{I}}$ is the $(N+1)^3\times(N+1)^3$ identity matrix and $W=[w_{i,j}],\quad i,j=1,2,\ldots,(N+1)^3.$

We now present the alternative forms for $u(x,y,t)$ which are important for simplifying matrix forms of the conditions. The simplification in conditions is done only with respect to the variables x, y and t. Therefore we must use different forms for initial and boundary conditions. For the initial conditions (2),

$$u(x,y,t)=\hat{B}(x,y)Q(t) \quad (13)$$

$$Q(t)=\left[B_0(t)\hat{I}\ \ B_1(t)\hat{I}\ \cdots B_N(t)\hat{I}\right]_{(N+1)^2\times(N+1)^3}$$

while for the boundary conditions (3),

$$u(x,y,t)=\hat{B}(x,t)Y^*(y) \quad (14)$$

$$\hat{B}(x,t)=\left[\hat{B}_0(x,t)\ \ \hat{B}_0(x,t)\cdots\hat{B}_N(x,t)\right]_{1\times(N+1)^2};$$

$$\hat{B}_i(x,t)=B_i(t)B(x)$$

$$Y(y)=\left[B_0(y)\hat{I}\ \ B_1(y)\hat{I}\cdots B_N(y)\hat{I}\right]_{(N+1)\times(N+1)^2},$$

$$Y^*(y)=\begin{bmatrix} Y(y) & 0 & \cdots & 0\\ 0 & Y(y) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & Y(y)\end{bmatrix}_{(N+1)^2\times(N+1)^3},$$

and also we use

$$u(x,y,t)=\hat{B}(y,t)L(x) \quad (15)$$

$$\hat{B}(y,t)=\left[\hat{B}_0(y,t)\ \ \hat{B}_1(y,t)\cdots\hat{B}_N(y,t)\right]_{1\times(N+1)^2},$$

$$\hat{B}_i(y,t)=B_i(t)B(y),$$

$$B(y)=\left[B_0(y)\ \ B_1(y)\cdots B_N(y)\right],$$

$$L(x)=\begin{bmatrix} B(x) & 0 & \cdots & 0\\ 0 & B(x) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & B(x)\end{bmatrix}_{(N+1)^2\times(N+1)^3}.$$

Notice also that the matrices involved in the right-hand side of Eqs. (2) and (3) are of the form

$$\psi_1(x,y)=B(x,y)\psi_1,$$

$$\psi_1=\left[\tau_{0,0}\ \ \tau_{0,1}\cdots\tau_{0,N}\ \tau_{1,0}\ \tau_{1,1}\cdots\tau_{1,N}\cdots\tau_{N,0}\ \tau_{N,1}\cdots\tau_{N,N}\right]^T, \quad (16)$$

$$\tau_{m,n}=\frac{1}{m!n!}\int_0^1\int_0^1\frac{\partial^{m+n}\psi_1(x,y)}{\partial x^m\partial y^n}dydx,$$

$$\psi_2(x,y)=B(x,y)\psi_2,$$

$$\psi_2=\left[\gamma_{0,0}\ \ \gamma_{0,1}\cdots\gamma_{0,N}\ \gamma_{1,0}\ \gamma_{1,1}\ \cdots\ \gamma_{1,N}\cdots\ \gamma_{N,0}\ \ \gamma_{N,1}\cdots\gamma_{N,N}\right]^T \quad (17)$$

$$\gamma_{m,n}=\frac{1}{m!n!}\int_0^1\int_0^1\frac{\partial^{m+n}\psi_2(x,y)}{\partial x^m\partial y^n}dydx,$$

$$f_0(x,t)=B(x,t)F_0,$$

$$F_0=\left[\mu_{0,0}\ \ \mu_{0,1}\cdots\mu_{0,N}\ \mu_{1,0}\ \ \mu_{1,1}\cdots\mu_{1,N}\cdots\mu_{N,0}\ \ \mu_{N,1}\cdots\mu_{N,N}\right]^T \quad (18)$$

$$\mu_{m,n}=\frac{1}{m!n!}\int_0^1\int_0^1\frac{\partial^{m+n}f_0(x,t)}{\partial x^m\partial t^n}dtdx,$$

$$f_1(x,t)=B(x,t)F_1,$$

$$F_1=\left[\xi_{0,0}\ \ \xi_{0,1}\cdots\xi_{0,N}\ \xi_{1,0}\ \ \xi_{1,1}\cdots\xi_{1,N}\cdots\xi_{N,0}\ \ \xi_{N,1}\cdots\xi_{N,N}\right]^T, \quad (19)$$

$$\xi_{m,n}=\frac{1}{m!n!}\int_0^1\int_0^1\frac{\partial^{m+n}f_1(x,t)}{\partial x^m\partial t^n}dtdx,$$

$$g_0(y,t)=B(y,t)G_0,$$

$$G_0=\begin{bmatrix}\zeta_{0,0}\ \ \zeta_{0,1}\cdots\zeta_{0,N}\ \zeta_{1,0}\ \zeta_{1,1}\cdots\\ \zeta_{1,N}\cdots\zeta_{N,0}\ \ \zeta_{N,1}\cdots\zeta_{N,N}\end{bmatrix}^T, \quad (20)$$

$$\zeta_{m,n}=\frac{1}{m!n!}\int_0^1\int_0^1\frac{\partial^{m+n}g_0(y,t)}{\partial y^m\partial t^n}dtdx,$$

$$g_1(y,t)=B(y,t)G_1,$$

$$G_1=\begin{bmatrix}\sigma_{0,0}\ \sigma_{0,1}\cdots\sigma_{0,N}\ \sigma_{1,0}\ \ \sigma_{1,1}\\ \cdots\sigma_{1,N}\cdots\sigma_{N,0}\ \ \sigma_{N,1}\cdots\sigma_{N,N}\end{bmatrix}^T, \quad (21)$$

$$\sigma_{m,n}=\frac{1}{m!n!}\int_0^1\int_0^1\frac{\partial^{m+n}g_1(y,t)}{\partial y^m\partial t^n}dtdx.$$

By substituting relations (16)–(21) into (2) and (3) and then simplifying the result, we get the matrix forms of the conditions, respectively, as

$$Q(0)A=\psi_1 \quad or \quad K_1A=\psi_1 \quad (22)$$

$$Q(0)SA=\psi_2 \quad or \quad K_2A=\psi_2 \quad (23)$$

$$Y^*(0)A=F_0 \quad or \quad K_3A=F_0 \quad (24)$$

$$Y^*(1)A=F_1 \quad or \quad K_4A=F_1 \quad (25)$$

$$\mathrm{L(0)A=G_0} \quad or \quad \mathrm{K_5A=G_0} \quad (26)$$

$$\mathrm{L(1)A=G_1} \quad or \quad \mathrm{K_6A=G_1} \quad (27)$$

$$\left[\hat{W};\hat{G}\right]=\begin{bmatrix}K_1 & ; & \psi_1\\ K_2 & ; & \psi_2\\ K_3 & ; & F_0\\ K_4 & ; & F_1\\ K_5 & ; & G_0\\ K_6 & ; & G_1\\ \bar{W} & ; & \bar{G}\end{bmatrix} \quad (28)$$

The unknown Bernoulli coefficients are obtained as

$$A=\left(\hat{\hat{W}}\right)^{-1}\hat{\hat{G}},$$

Where $[\hat{\hat{W}};\hat{\hat{G}}]$ is generated by using the Gauss elimination method and then removing zero rows of the Gauss eliminated matrix. Here $\bar{W}$ and $\bar{G}$ are obtained by throwing away the maximum number of row vectors from W and G, so the rank of the system defined in (28) cannot be smaller than $(N+1)^3$. This process provides higher accuracy because of the decreasing truncation error.

3. Accuracy of the Solution and Error Analysis

We can easily check the accuracy of the method. Since the truncated Bernoulli series (4) is an approximate solution ofEq. (1), when the function $u_{N,N,N}(x,y,t)$ and its derivatives are substituted in Eq. (1), the resulting equation must be satisfiedapproximately; that is, for

$$(x,y,t)=(x_p,y_q,t_r)\in[0,1]\times[0,1]\times(0,T),\quad p,q,r=0,1,2,$$

$$E_{N,N,N}(x_p, y_q, t_r) = \left| \frac{\partial^2 u}{\partial t^2}(x_p, y_q, t_r) \right.$$
$$+2\alpha \frac{\partial^u}{\partial t}(x_p, y_q, t_r) + \beta^2 u(x_p, y_q, t_r)$$
$$\left. +\theta \frac{\partial^2 u}{\partial x^2}(x_p, y_q, t_r) + \eta \frac{\partial^2 u}{\partial y^2}(x_p, y_q, t_r) - f(x_p, y_q, t_r) \right| \approx 0.$$

and $E_{N,N,N}(x_p, y_q, t_r) \le 10^{-k_{pqr}}$ (k_{pqr} positive integer). If max $10^{-k_{pqr}} = 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{N,N,N}(x_p, y_q, t_r)$ at each of the points becomes smaller than the prescribed 10^{-k} .

On the other hand, the error, can be estimated by L_∞ and L_2 errors, and the root mean square error (RMS). We calculate the RMS error using the following formula :

$$RMS \ \ error = \sqrt{\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(u(x_i, y_j, \tau) - \hat{u}(x_i, y_j, \tau)\right)^2}{mn}},$$

where u and $\hat{u}$ are the exact and approximate solutions of the problem, respectively, and τ is an arbitrary time t in $[0,1]$.

4. IllustrativeExample

This section is devoted to a computational result. We applied the method presented in this work and solved one example. We illustrate our work with the following example. The numerical computations were done using MATLAB v7.12.0.

Example.Consider the problem

$$u_{tt} + 2u_t + u - u_{xx} - u_{yy} = -2 + x^2 + y^2 + t.$$

The initial conditions are given by

$$u(x, y, 0) = x^2 + y^2,$$
$$u_t(x, y, 0) = x^2 + y^2 + 1,$$

By applying the technique described in the preceding section we find the matrix representation of the equation as follows:

$$\left\{ (S)^2 + 2S + \hat{\hat{I}} - \left(\hat{\hat{H}}\right)^2 - \left(\hat{M}\right)^2 \right\} A = F.$$

Solving the above linear system yields to

$$u(x, y, t) = x^2 + y^2 + t$$

which is the exact solution.

5. Conclusions

Two-space-dimensional linear hyperbolic equations with constant coefficients are usually difficult to solve analytically. For this purpose, the present method has been proposed for approximate solutions and also analytical solutions. It is observed that the method has a great advantage when the known functions in the equation can be expanded in Bernoulli series. Moreover, this method is applicable for the approximate solution of parabolic and elliptic type partial differential equations in two space dimensions. The method can also be extended to nonlinear partial differential equations, but some modifications are required.

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