

Application of He's Variational Iteration Method for Solving Seventh-Order Differential Equations

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Abstract In this paper, we extend variational iteration method (VIM) for deriving approximate analytical solution to seventh-order differential equations with specified initial conditions, also in this paper we applied a modified method to identification of Lagrange multiplier. By providing some examples, we illustrate the capability and reliability of the method.

Keywords Variational Iteration Method (VIM); Seventh-Order Equations; Analytical Solutions

1. Introduction

Recently, many different methods have been introduced to solve linear and nonlinear problems, such as the homotopy analysis method[1], the variational iteration method (VIM)[2-5], Energy Balance Method[6], the Adomian's decomposition method (ADM)[7,8], and homotopy perturbation method[9-12].

Ji-Huan. He proposed a variational iteration method (VIM) based on the use of restricted variations and correction functional which has found a wide application for the solution of nonlinear ordinary and partial differential equations[13-15]. VIM has successfully been applied to many situations. For example, Batiha et al.[16] applied VIM to solve heat- and wave-like equations with singular behaviors. Furthermore, Batiha et al.[17] have expanded VIM in the form of Multistage VIM to solve a class of nonlinear system of ODEs, Wazwaz[18] applied VIM to solve linear and nonlinear Schrodinger equations. Shou et al.[19] solved heat-like and wave-like equations with variable coefficients by VIM, Sweilam[20] used VIM to solve multi-order FDEs. The VIM is capable for solving a large class of linear or nonlinear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also it reduces the size of calculations. In this paper, we implement the VIM for finding the approximate analytical solutions of nonlinear seventh-order equations with specified initial conditions.

In the present paper we employ VIM method for solving following equation:

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial^7}{\partial x^7}u(x,t) = F(x,t,u^2, \frac{\partial^2}{\partial x^2}u(x,t), \frac{\partial^3}{\partial x^3}u(x,t), \frac{\partial^4}{\partial x^4}u(x,t), \frac{\partial^5}{\partial x^5}u(x,t)), \quad (1)$$

but first we introduce VIM method and then we extend this method.

2. Analysis of the Variational Iteration Method (VIM)

To illustrate the basic concept of the technique, we consider the following general differential equation between columns.

$$Lu + Nu = g(x), \quad (2)$$

Where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to He's variational iteration method[21-23], He and Wu[24], Inokuti et al.[25], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + Nu_n(s) - g(s))ds \quad (3)$$

Where λ is a Lagrange multiplier. In this paper, we studied about modifying the traditional method to achieve λ with the fewer steps which would be much easier and faster than the previous accomplishment. We came to the modified method in which we do not need to calculate with hands and in order to obtain λ , it is sufficient to locate the linear part of differential equation at the disclosed program and use Laplace transform for linear part of differential equation following with a few simple assumption, we will catch the accurate results. The successive approximation u_{n+1} , $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function, u_0 consequently, the solution is given by

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$$u = \lim_{n \rightarrow \infty} u_n \tag{4}$$

3. Application of the Variational Iteration Method (VIM)

In this section, we will apply the variational iteration method (VIM) to solve following equations:

3.1. Example 1

Consider the following differential equations of the seventh order:

$$\begin{aligned} & \frac{\partial}{\partial t} u(x, t) + \frac{\partial^7}{\partial x^7} u(x, t) + 30 \frac{\partial^2}{\partial x^2} u(x, t) \frac{\partial^3}{\partial x^3} u(x, t) + \\ & 60 \frac{\partial}{\partial x} u(x, t) \frac{\partial^4}{\partial x^4} u(x, t) + 30u(x, t) \frac{\partial^5}{\partial x^5} u(x, t) + \\ & 360u(x, t) \left(\frac{\partial}{\partial t} u(x, t)\right)^2 + 180u(x, t)^2 \frac{\partial^2}{\partial x^2} u(x, t) = 0, \end{aligned} \tag{5}$$

With the initial condition

$$u_0(x, t) = \mu^2 \operatorname{sech}(\mu x)^2 \tag{6}$$

To solve Eq. (5) by means of VIM, we have the correction functional as:

$$\begin{aligned} u_{n+1}(x, t) = & u_n(x, t) + \int_0^t \lambda \left(\frac{\partial}{\partial \tau} u(x, \tau) + \right. \\ & \left. \frac{\partial^7}{\partial x^7} u(x, \tau) + 30 \frac{\partial^2}{\partial x^2} u(x, \tau) \frac{\partial^3}{\partial x^3} u(x, \tau) + \right. \\ & \left. 60 \frac{\partial}{\partial x} u(x, \tau) \frac{\partial^4}{\partial x^4} u(x, \tau) + 30u(x, \tau) \frac{\partial^5}{\partial x^5} u(x, \tau) + \right. \\ & \left. 360u(x, \tau) \left(\frac{\partial}{\partial t} u(x, \tau)\right)^2 + 180u(x, \tau)^2 \frac{\partial^2}{\partial x^2} u(x, \tau) \right) d\tau \end{aligned} \tag{7}$$

We apply Laplace transform for linear part of the Eq. (5). Linear part of equation is:

$$u_t \tag{8}$$

To find the optimal values of λ we have

$$sU(s) - u(0) = (-1)^n, n = 1 \tag{9}$$

And with assumption that boundary conditions is zero

$$u(0) = 0 \tag{10}$$

Substituting Eq. (10) into Eq. (9), we have:

$$\begin{cases} sU(s) = -1 \\ U(s) = \frac{-1}{s} \\ u(t) = -1 \end{cases} \tag{11}$$

Lagrange multiplier can be identified optimally as follows:

$$\lambda = -1 \tag{12}$$

As a result, we obtain the following iteration formula:

$$\begin{aligned} u_{n+1}(x, t) = & u_n(x, t) - \int_0^t \left(\frac{\partial}{\partial \tau} u(x, \tau) + \frac{\partial^7}{\partial x^7} u(x, \tau) + \right. \\ & 30 \frac{\partial^2}{\partial x^2} u(x, \tau) \frac{\partial^3}{\partial x^3} u(x, \tau) + 60 \frac{\partial}{\partial x} u(x, \tau) \frac{\partial^4}{\partial x^4} u(x, \tau) + \\ & 30u(x, \tau) \frac{\partial^5}{\partial x^5} u(x, \tau) + 360u(x, \tau) \left(\frac{\partial}{\partial t} u(x, \tau)\right)^2 + \\ & \left. 180u(x, \tau)^2 \frac{\partial^2}{\partial x^2} u(x, \tau) \right) d\tau \end{aligned} \tag{13}$$

To get the iteration, we start with an initial approximation, that was given by Eq. (6). By using the above iteration formula (13) we can obtain the other components by using mathematical tools MAPLE package as follows:

$$u_0(x, t) = \mu^2 \operatorname{sech}(\mu x)^2 \tag{14}$$

$$\begin{aligned} u_1(x, t) = & \frac{1}{\cosh(\mu x)^8} (\mu^2 \cosh(\mu x)^6 + \\ & 128\mu^7 \sinh(\mu x)t \cosh(\mu x)^5 - 4224\mu^7 \sinh(\mu x)t \cosh(\mu x)^3 + \\ & 7200\mu^7 \sinh(\mu x)t \cosh(\mu x) - 2160\mu^6 t \cosh(\mu x)^2 + 2520\mu^6 t) \end{aligned} \tag{15}$$

$$\begin{aligned} u_2(x, t) = & \frac{1}{\cosh(\mu x)^{11}} (\mu^2 (6512640\mu^{16}t^3 \cosh(\mu x)^3 + \\ & 58982400\mu^{21} \sinh(\mu x)t^4 \cosh(\mu x)^2 + \cosh(\mu x)^9 - \\ & 24576000\mu^{17} \sinh(\mu x)t^3 \cosh(\mu x)^4 + 5040\mu^6 t \cosh(\mu x)^3 - \\ & 8448\mu^7 \sinh(\mu x)t \cosh(\mu x)^6 + 14400\mu^7 \sinh(\mu x)t \cosh(\mu x)^4 + \\ & 73728000\mu^{17} \sinh(\mu x)t^3 \cosh(\mu x)^2 + \\ & 161280\mu^{11} \sinh(\mu x)t^2 \cosh(\mu x)^2 - 55296000\mu^{17} \sinh(\mu x)t^3 - \\ & 4423680\mu^{16}t^3 \cosh(\mu x) + 514560\mu^{12}t^2 \cosh(\mu x)^5 - \\ & 403200\mu^{12}t^2 \cosh(\mu x)^3 - 32\mu^5 \sinh(\mu x)t \cosh(\mu x)^8 - \\ & 44236800\mu^{21} \sinh(\mu x)t^4 - 17694720\mu^{21} \sinh(\mu x)t^4 \cosh(\mu x)^4 + \\ & 1310720\mu^{17} \sinh(\mu x)t^3 \cosh(\mu x)^6 - \\ & 103680\mu^{11} \sinh(\mu x)t^2 \cosh(\mu x)^4 - 4320\mu^6 t \cosh(\mu x)^5 - \\ & 2211840\mu^{16}t^3 \cosh(\mu x)^5 + 2048\mu^{12}t^2 \cosh(\mu x)^9 - \\ & 138240\mu^{12}t^2 \cosh(\mu x)^7 + 256\mu^7 \sinh(\mu x)t \cosh(\mu x)^8)) \end{aligned} \tag{16}$$

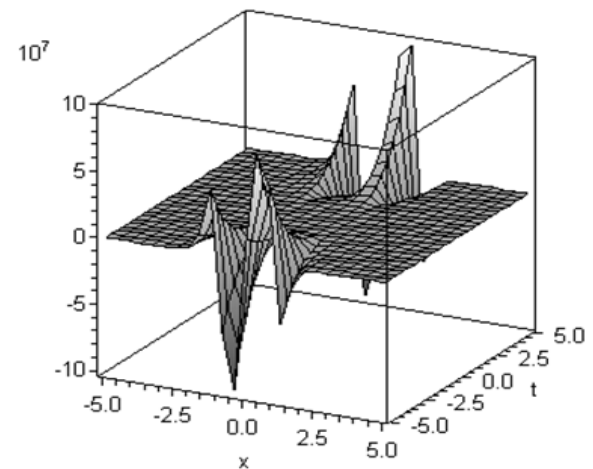


Figure 1. The surface generated from $u(x, t)$ of variational iteration method (VIM) for seventh-order equation when $\mu=0.1$

Table 1. Numerical results for example 1 when $\mu=0.1$

variational iteration method			
X_i	$t=0$	$t=1$	$t=2$
0.1	0.9900662910	-8.007203128 10 ⁵	-8.407564605 10 ⁶
0.2	0.9610429822	-9.644522044 10 ⁵	-1.032605959 10 ⁷
0.3	0.9151369621	-5.710340821 10 ⁵	-5.973163748 10 ⁶
0.4	0.8556387858	0.790583128 10 ⁵	1.301522958 10 ⁶
0.5	0.7864477333	5.982850695 10 ⁵	7.106618340 10 ⁶
0.6	0.7115777629	7.797146068 10 ⁵	9.078708613 10 ⁶
0.7	0.6347395896	6.545862198 10 ⁵	7.565921749 10 ⁶
0.8	0.5590551680	3.811664740 10 ⁵	4.372763372 10 ⁶
0.9	0.4869173615	1.114107127 10 ⁵	1.237156006 10 ⁶
1	0.4199743415	-0.741602391 10 ⁵	-9.18560812 10 ⁵

3.2. Example 2

Next, we consider the following equation:

$$\begin{aligned} &\frac{\partial}{\partial t}u(x,t) + \frac{\partial^7}{\partial x^7}u(x,t) + 63\frac{\partial^2}{\partial x^2}u(x,t)\frac{\partial^3}{\partial x^3}u(x,t) + \\ &42\frac{\partial}{\partial x}u(x,t)\frac{\partial^4}{\partial x^4}u(x,t) + 21u(x,t)\frac{\partial^5}{\partial x^5}u(x,t) + \\ &378u(x,t)\frac{\partial^2}{\partial x^2}u(x,t)\frac{\partial}{\partial x}u(x,t) + 126u(x,t)^2\frac{\partial^3}{\partial x^3}u(x,t) + \\ &252u(x,t)^3\frac{\partial}{\partial x}u(x,t) + 63\left(\frac{\partial}{\partial x}u(x,t)\right)^3 = 0 \end{aligned} \tag{17}$$

With the initial condition

$$u_0(x,t) = \frac{4}{3}\mu^2(2 - 3 \tanh(\mu x)^2) \tag{18}$$

To solve Eq. (17) by means of VIM, we have the correction functional as:

$$\begin{aligned} u_{n+1}(x,t) = &u_n(x,t) + \int_0^t \lambda \left(\frac{\partial}{\partial t}u(x,\tau) + \frac{\partial^7}{\partial x^7}u(x,\tau) + \right. \\ &63\frac{\partial^2}{\partial x^2}u(x,\tau)\frac{\partial^3}{\partial x^3}u(x,\tau) + 42\frac{\partial}{\partial x}u(x,\tau)\frac{\partial^4}{\partial x^4}u(x,\tau) + \\ &21u(x,\tau)\frac{\partial^5}{\partial x^5}u(x,\tau) + 378u(x,\tau)\frac{\partial^2}{\partial x^2}u(x,\tau)\frac{\partial}{\partial x}u(x,\tau) + \\ &126u(x,\tau)^2\frac{\partial^3}{\partial x^3}u(x,\tau) + 252u(x,\tau)^3\frac{\partial}{\partial x}u(x,\tau) \\ &\left. + 63\left(\frac{\partial}{\partial x}u(x,\tau)\right)^3 \right) d\tau \end{aligned} \tag{19}$$

We apply Laplace transform for linear part of the Eq. (17). Linear part of equation is:

$$u_t \tag{20}$$

To find the optimal values of λ we have

$$sU(s) - u(0) = (-1)^n, n = 1 \tag{21}$$

And with assumption that boundary conditions is zero

$$u(0) = 0 \tag{22}$$

Substituting Eq. (22) into Eq. (21), we have:

$$\begin{cases} sU(s) = -1 \\ U(s) = \frac{-1}{s} \\ u(t) = -1 \end{cases} \tag{23}$$

Lagrange multiplier can be identified optimally as follows:

$$\lambda = -1 \tag{24}$$

As a result, we obtain the following iteration formula:

$$\begin{aligned} u_{n+1}(x,t) = &u_n(x,t) - \int_0^t \left(\frac{\partial}{\partial t}u(x,\tau) + \frac{\partial^7}{\partial x^7}u(x,\tau) + \right. \\ &63\frac{\partial^2}{\partial x^2}u(x,\tau)\frac{\partial^3}{\partial x^3}u(x,\tau) + 42\frac{\partial}{\partial x}u(x,\tau)\frac{\partial^4}{\partial x^4}u(x,\tau) + \\ &21u(x,\tau)\frac{\partial^5}{\partial x^5}u(x,\tau) + 378u(x,\tau)\frac{\partial^2}{\partial x^2}u(x,\tau)\frac{\partial}{\partial x}u(x,\tau) + \\ &126u(x,\tau)^2\frac{\partial^3}{\partial x^3}u(x,\tau) + 252u(x,\tau)^3\frac{\partial}{\partial x}u(x,\tau) + \\ &\left. 63\left(\frac{\partial}{\partial x}u(x,\tau)\right)^3 \right) d\tau \end{aligned} \tag{25}$$

To get the iteration, we start with an initial approximation that was given by Eq. (18). By using the above iteration

formula (25) we can obtain the other components by using mathematical tools MAPLE package as follows:

$$u_0(x,t) = \frac{4}{3}\mu^2(2 - 3 \tanh(\mu x)^2) \tag{26}$$

$$u_1(x,t) = -\frac{4}{3}\frac{1}{\cosh(\mu x)^3}(\mu^2(512\mu^7 \sinh(\mu x)t + \cosh(\mu x)^3 - 3 \cosh(\mu x))) \tag{27}$$

$$\begin{aligned} u_2(x,t) = &-\frac{4}{9}\frac{1}{\cosh(\mu x)^{13}}(\mu^2(1803886264320\mu^{28}t^4 \cosh(\mu x)^7 \\ &- 262144\mu^{14}t^2 \cosh(\mu x)^{11} - 9 \cosh(\mu x)^7 t^4 - \\ &8433168285696\mu^{28}t^4 \cosh(\mu x) + 3 \cosh(\mu x)^{13} + \\ &3072\mu^7 \sinh(\mu x)t \cosh(\mu x)^{10} + 393216\mu^{14}t^2 \cosh(\mu x)^9 - \\ &369435906932736\mu^{35}t^5 \sinh(\mu x) + \\ &16866336571392\mu^{28}t^4 \cosh(\mu x)^3 - \\ &10282151706624\mu^{28}t^4 \cosh(\mu x)^5 + \\ &615726511554560\mu^{35} \sinh(\mu x)t^5 \cosh(\mu x)^2 - \\ &246290604621824\mu^{35} \sinh(\mu x)t^5 \cosh(\mu x)^4)) \end{aligned} \tag{28}$$

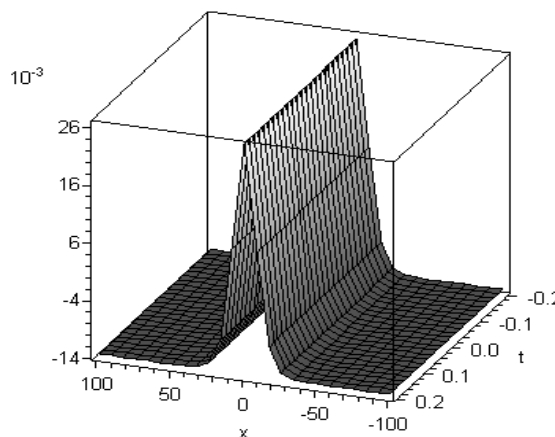


Figure 2. The surface generated from $u(x, t)$ of variational iteration method (VIM) for seventh-order equation when $\mu=0.1$

Table 2. Numerical results for example 2 when $\mu=0.1$

variational iteration method			
X_i	$t=0$	$t=1$	$t=2$
0.1	0.02666266698	0.02666266333	0.02666263966
0.2	0.02665067091	0.02665064361	0.02665061629
0.3	0.02663068824	0.02663064733	0.02663060640
0.4	0.02660273485	0.02660268034	0.02660262583
0.5	0.02656683313	0.02656676512	0.02656669706
0.6	0.02652301157	0.02652293003	0.02652284849
0.7	0.02647130514	0.02647121017	0.02647111520
0.8	0.02641175499	0.02641164666	0.02641153834
0.9	0.02634440830	0.02634428675	0.02634416516
1	0.02626931831	0.02626918357	0.02626904883

4. Conclusions

In the present work, illustrative examples have showed that the method provides a sequence of functions which converge to the exact solution of the aforementioned systems. The results variational iteration method (VIM) are of high concentration and the method is very effective and succinct.

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